Sets, Relations and Functions

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August 21, 2015

1 Basic Set Theory

We often groups things together. Everyone in this class, your group of friends, your family. These are all collections of people. Set theory is a mathematical language to talk about collections.

The easiest way to visualize a set is to use a *Venn diagram.* A Venn Diagram is a geometrical visualization of a set, or collection of sets. Abstractly, a set is just a collection of objects that have something in common. We will denote sets with upper-case letters, and elements of sets will be denoted with lower-case letters. There are two ways to write down the contents of a set:

1. List all the elements of the set. Each element should be separated by a comma and there entire list of elements is written between curly brackets: $\{$ and $\}$. For example suppose A is the set of the first 5 letters of the alphabet. Then $A = \{a, b, c, d, e\}$.

2. Write down a property that *all* elements of the set have in common. For example, if A is the set of all positive integers, then we can describe A as follows $A = \{x \mid x \geq 0 \text{ and } x \text{ is an integer}\}$. This is read "$A$ is the set of all x such that x is an integer that is greater than or equal to zero".

Definition 1.1 (Set) Any collection of objects (formally, a set is a collection of elements of a *universal set*[1]). $\triangleleft$

[1] Typically, we start by fixing a universal set that defines the "domain of discourse".

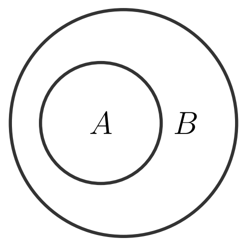
Definition 1.2 (Element) A member of a set. ◁

We write $x \in A$ to mean "x is an element of the set A".

Definition 1.3 (Subset) A set A is a **subset** of a set B, denoted $A \subseteq B$, provided every element of A is also an element of B. ◁

Notice that every set is a subset of the universal set. The notion of subset can be pictured as follows:

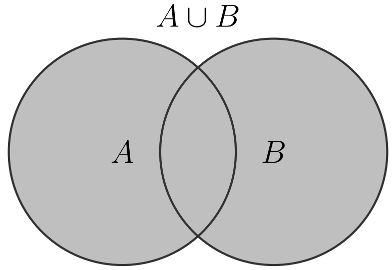
$A \subseteq B$



Definition 1.4 (Union) The **union** of two (or more) sets is a set that contains all the elements of each set. For two sets A and B, the union of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. ◁

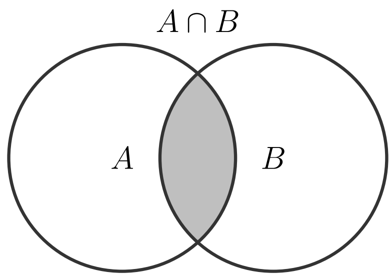
The union of two sets can be pictured as follows (the gray shaded region is the union of A and B):

$A \cup B$



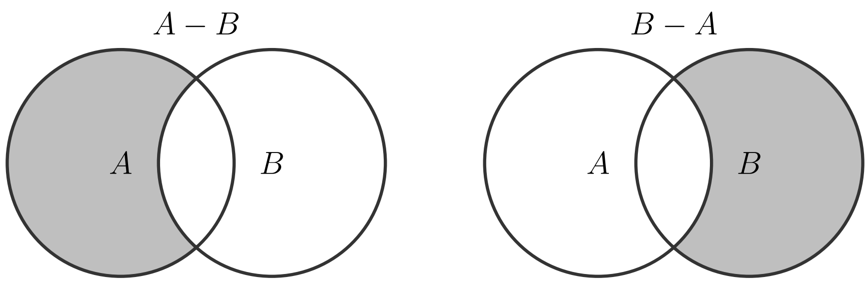
Definition 1.5 (Intersection) The **intersection** of two (or more) sets is the set of all items in common to each set. If A and B are two sets, then the intersection of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. ◁

The intersection can be pictured as follows (the gray shaded region is the intersection of A and B):



Definition 1.6 (Set Difference) The difference between two sets A and B (A minus B) is all elements in A but not in B. The difference between A and B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$. $\triangleleft$

The differences between A and B can be pictured as follows:



Definition 1.7 (Complement) The **Complement** of a set is the set of all elements not contained in that set. Formally, the complement of the set A is $A^C = \{x \mid x \text{ is in the universal set and } x \notin A\}$ $\triangleleft$

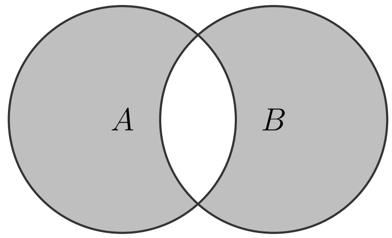
Why is the notion of a universal set necessary for this definition?

Exercise 1.8 *Using Venn diagrams, convince yourself that for any sets A and B, $A - B = A \cap B^C$*

Definition 1.9 (Symmetric Difference) The symmetric difference of two sets is all the elements in either set *but not in both*. The symmetric difference is the set $(A - B) \cup (B - A)$. $\triangleleft$

The symmetric difference can be pictured as follows:

$(A - B) \cup (B - A)$



Definition 1.10 (Null Set) The null or empty set is a set that contains no elements. We write $\emptyset$ to denote the set containing no elements. $\triangleleft$

Definition 1.11 (Power Set) The **power set** is the set of all subsets. If A is a set, the power set of A is the set $\wp(A) = \{B \mid B \subseteq A\}$. $\triangleleft$

Notice that the empty set is a subset of every set.

Definition 1.12 (Cardinality of a Set) The cardinality of a finite[2] set A is the total number of elements in A, and is denoted $|A|$. $\triangleleft$

Definition 1.13 (Partition) A **partition** of a set S is a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots\}$ (possibly infinite) such that

- the sets are **pairwise disjoint**: if $S_i, S_j \in \mathcal{S}$ with $i \neq j$, then $S_i \cap S_j = \emptyset$
- their union is S, that is, $S = \cup_{S_i \in \mathcal{S}} S_i$. $\triangleleft$

2 Relations

Suppose that X is a non-empty set. The set $X \times X$ is the **cross-product** of X with itself. That is, it is the set of all pairs of elements (called **ordered pairs**) from X. For example, if $X = \{a, b, c\}$, then

$$X \times X = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$$

A **relation** R on a set X is a subset of $X \times X$ (the set of pairs of elements from X). Formally, R is a relation on X means that $R \subseteq X \times X$. It is often convenient to write $a\ R\ b$ for $(a, b) \in R$. To help appreciate this definition, consider the following example. Suppose that X is the set of people in a room.

[2]The notion of cardinality can be applied to infinite sets as well. However, a discussion of this is beyond the scope of these introductory notes.

Further, suppose that everyone in the room is pointing at some person in the room. A relation can be used to describe who is pointing at whom, where for $a, b \in X$, $a\ R\ b$ means that person a is pointing at person b. A second example of a relation is "taller-than", denoted $T \subseteq X \times X$, where $a\ T\ b$ means that person a is taller than person b. Typically, we are interested in relations satisfying special properties.

Definition 2.1 (Key Properties of relations) Suppose that $R \subseteq X \times X$ is a relation.

- R is **reflexive** provided for all $a \in X$, $a\ R\ a$
- R is **irreflexive** provided for all $a \in X$, it is not the case that $a\ R\ a$
- R is **complete** provided for all $a, b \in X$, $a\ R\ b$ or $b\ R\ a$ (or both)
- R is **symmetric** provided for all $a, b \in X$, if $a\ R\ b$ then $b\ R\ a$
- R is **asymmetric** provided for all $a, b \in X$, if aRb then not-$b\ R\ a$
- R is **transitive** provided for all $a, b, c \in X$, if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$

$\triangleleft$

Remark 2.2 *As stated, completeness implies reflexivity (let $a = b$ in the above statement). Often, one states completeness as follows: for all* distinct *$a, b \in X$, aRb or bRa. In what follows, we will use the above stronger definition of completeness where completeness implies reflexivity.*

Recall the example of a relation R that describes people pointing at other people in the room. If R is reflexive, then this means everyone is pointing at themselves. If R is irreflexive, then this means that no-one is pointing at themselves. This example illustrates the fact that irrelexivity is *not* the negation of reflexivity. That is, there are examples of relations that are neither reflexive nor irreflexive. If R is complete, then this means that every person in the room is either pointing at somebody or being pointed at. Symmetry of R means that every person that is being pointed at is pointing back at the person pointing at them. Asymmetry of R means that nobody is pointing back at the person pointing at them. Similar to the relationship between reflexivity and irreflexivity, asymmetry is *not* the negation of symmetry. Finally, picturing transitivity of the relation R is a bit more complicated. If the relation R is transitive, then everyone is pointing at the person that is being pointed to by the person that they are pointing at.

Exercise 2.3 *Suppose that there are 5 people in a room. Draw a picture of a situation where the people are pointing at each other and the relation that describes the situation is transitive.*

Exercise 2.4 *What properties does the "better-than" relations satisfy?*

Remark 2.5 (Describing Relations) *Suppose that $R \subseteq X \times X$ is a relation. We will often use the following shorthand to denote elements in the relation: If $x_1, \ldots, x_n \in X$, then*

$$x_1 \; R \; x_2 \; R \cdots x_{n-1} \; R \; x_n$$

means that for all $i = 1, \ldots, n-1$, $(x_i, x_{i+1}) \in R$ or $(x_i, x_j) \in R$ for all $j < i$ if R is assumed to be transitive (or $j \leq i$ if R is assumed to also be reflexive). For example, if R is transitive and reflexive, then $a \; R \; b \; R \; c$ means that $(a, a), (a, b), (b, b), (a, c), (b, c), (c, c) \in R$.

The following two definitions will play an important role in this course.

Definition 2.6 (Cycle) A **cycle** in a relation $R \subseteq X \times X$ is a set of distinct elements $x_1, x_2, \ldots, x_n \in X$ such that for all $i = 1, \ldots, n-1$, $x_i \; R \; x_{i+1}$, and $x_n \; R \; x_1$. A relation R is said to be **acyclic** if there is no cycles. $\triangleleft$

Definition 2.7 (Maximal Elements) Suppose that X is a set and $S \subset X$. An element $a \in S$ is **maximal** provided there is no $b \in S$ such that $b \; R \; a$. Let $\max_R(S)$ be the set of maximal elements of Y. $\triangleleft$

Exercise 2.8 *Suppose that X has three elements (i.e., $X = \{a, b, c\}$. How many cycles can be formed from elements in X?*

Exercise 2.9 *Is it possible to find a relation that has a cycle and a non-empty set of maximal elements? What about a relation that has a cycle, a non-empty set of maximal elements, and is complete and transitive?*

Exercise 2.10 *Prove that if R is acyclic, then $\max_R(Y) \neq \emptyset$. Is the converse true? (Why or why not?)*

Relations are an important mathematical tool used throughout Economics, Logic and Philosophy. You have already studied binary relations during your mathematical eduction: $=, \leq, \geq, <$, and $>$ are all relations on numbers (eg., the natural numbers $\mathbb{N}$, real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$, etc.) and $\subseteq$ is a relation on the power set of a set S. For example, the binary relation $\leq \subseteq \mathbb{N} \times \mathbb{N}$ is the set

$$\{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \text{ is less than or equal to } b\}$$

Definition 2.11 (Equivalence Relation) A relation R that is reflexive, symmetric and transitive is said to be an **equivalence relation** $\triangleleft$

Definition 2.12 (Equivalence Class) If R is an equivalence relation on A, then for each $a \in A$, the equivalence class of a, denoted by $[a]$, is the following set $[a] = \{b \mid aRb\}$. $\triangleleft$

Definition 2.13 (Partial Order) A relation that is reflexive, antisymmetric and transitive is said to be a **partial order**. $\triangleleft$

The standard example of a partial order is the relation $\subseteq$.

The following is our first theorem. It is somewhat technical, but illustrates a fundamental idea about equivalence classes and partitions. Namely, that every partition has an equivalence relation associated with it, and every equivalence class has a partition associated with it.

Theorem 2.14 *The equivalence classes of any equivalence relation R on a set A forms a partition of A, and any partition of A determines an equivalence relation on A for which the sets in the partition are the equivalence classes.*

Proof. Suppose R is an equivalence relation on A. We must show that the equivalence classes of R forms a partition of A.

1. Each equivalence class is non-empty, since $a \; R \; a$ for all $a \in A$.
2. Clearly A is the union of all the equivalence classes (since each element of A belongs to at least one equivalence class)
3. We must show any two equivalence classes are disjoint. Let $[a], [b]$ be two distinct equivalence classes. Suppose $c \in [a] \cap [b]$. Then $a \; R \; c$ and $b \; R \; c$. Hence by symmetry, $c \; R \; b$. And so by transitivity, $a \; R \; b$.

   Let $x \in [a]$, then $x \; R \; c$ and by the above argument $x \; R \; b$ (Why?), and so $x \in [b]$. Thus $[a] \subseteq [b]$. Using a similar argument, we can show $[b] \subseteq [a]$. Therefore $[a] = [b]$, which contradicts the fact that $[a]$ and $[b]$ are *distinct* equivalence classes.

For the second part of the theorem, suppose $\mathcal{A} = \{A_1, \ldots, A_n\}$ is any partition of A. Define $R = \{(a, b) \mid a \in A_i \text{ and } b \in A_i\}$. We must show that R is reflexive. Let $a \in A$ be any element. Then $a \in A_i$ for some i, and hence by definition of R,

$a \; R \; a$. Next we will show that R is symmetric. Suppose $a \; R \; b$. Then $a \in A_i$ and $b \in A_i$ for some i. Then clearly, $b \in A_i$ and $a \in A_i$ and hence $b \; R \; a$. We must show R is transitive. Suppose, $a \; R \; b$ and $b \; R \; c$. Then $a \in A_i$ and $b \in A_i$, and $b \in A_j$ and $c \in A_j$ for some i, j. Since $b \in A_i \cap A_j$, $A_i = A_j$ (since the elements of $\mathcal{A}$ are pairwise disjoint). Therefore, $a \in A_i$ and $c \in A_i$ and hence $a \; R \; c$. QED

Functions are a special type of relation:

Definition 2.15 (Function) A function f is a binary relation on A and B (i.e., $f \subseteq A \times B$) such that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. We write $f : A \to B$ when f is a function, and if $(a, b) \in f$, then write $f(a) = b$. $\triangleleft$

Suppose $f : A \to B$ is a function. A is said to be the **domain** and B the **codomain**.

Definition 2.16 (Image) The image of a set $A' \subseteq A$ is the set:

$$f(A') = \{b \mid b = f(a) \text{ for some } a \in A'\} \qquad \triangleleft$$

Definition 2.17 (Range) The **range** of a function is the image of its domain. $\triangleleft$

Suppose that $f : A \to B$ is a function.

Definition 2.18 (Surjection) f is a surjection (or onto) if its range is equal to its codomain. I.e., f is surjective iff for each $b \in B$, there exists an $a \in A$ such that $f(a) = b$ $\triangleleft$

Definition 2.19 (Injection) f is an injection (or 1-1) if distinct elements of the domain produce distinct elements of the codomain. I.e., f is 1-1 iff $a \neq a'$ implies $f(a) \neq f(a')$, or equivalently $f(a) = f(a')$ implies $a = a'$. $\triangleleft$

Definition 2.20 (Bijection) f is a bijection if it is injective and surjective. In this case, f is often called a one-to-one correspondence. $\triangleleft$

Definition 2.21 (Inverse Image) Suppose that $f : A \to B$ and that $Y \subseteq B$. The inverse image of Y is the set $f^{-1}(Y) = \{x \mid x \in A \text{ and } f(x) \in Y\}$ $\triangleleft$

3 Proofs

3.1 Introduction

Learning how to write mathematical proofs takes time and lots of practice. A proof of a mathematical statement is simply an explanation of that statement *written in the language of mathematics*. It is very important that you become comfortable with the definitions. If you don't know or understand the formal definitions, then you will not be able to write down your explanations. It would be like trying to explain something to someone in Italian without actually knowing Italian.

3.2 Proving Equality and Subset

How do you prove that two sets are equal? The answer to this question depends on who you are trying to convince. In this class, we will always err on the side of caution and given fairly detailed formal proofs. In turns out that proving two sets are equal reduces to proving the sets are subsets of each other.

Fact 3.1 *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$*

Why is this true? Well, if A and B are equal, then they both name the same collection of objects. I.e., B is another name for the collection of objects that A names and vice versa. So, if A and B are equal then of course $A \subseteq B$ since A is always a subset of itself and B is simply another name for A. Similarly, we can show $B \subseteq A$. Conversely, suppose $A \subseteq B$ and $B \subseteq A$. We want to know that A and B name the same collection of objects. Suppose they didn't, then there should be some object $x \in A$ that is not in B *or* some object $y \in B$ that is not in A. Well, we know such an object x cannot exist since $A \subseteq B$, and so, every element of A is an element of B. Similarly, the element y cannot exist. Hence, A and B must name the same collection of objects.

What about trying to prove that two sets are *not* equal? This turns out to be easier. In order to show that A does not equal B, you need only find an element in A that is not in B **OR** an element of B that is not in A.

Showing two sets are equal reduces to proving that the sets are subsets of each other. But, how to show that a set is a subset of another set? The general procedure to show $A \subseteq B$ is to show that each element of A is also and element of B. This is straightforward if A and B are both finite sets. For example, suppose $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. How do we show that $A \subseteq B$. Since A is finite, we simply notice that $2 \in B$, $3 \in B$ and $4 \in B$.

What if A is the set of even numbers and B is the set of all integers? We would get awfully tired (and bored) if we waited around to show that each and every element of A is also an element of B. Imagine A and B are two boxes, and you would like to know whether all the elements in A's box are also in B's box. Suppose you reach in box A and select an element, say the number 10. After inspecting 10, you notice that 10 is in fact an integer and so must also be an element of box B. But you are not satisfied, since you cannot be sure that the next element you choose from A will also be an element of B. In fact, even if you have shown that the first million even integers are all members of box B, you cannot be sure that the next element you select from box A will in fact be an integer. Instead, you should consider the *property* that all elements of A have in common and show that any object satisfying that property must be an element of B. What property does x satisfy if it is contained in A's box? The answer is $x = 2 \cdot n$, where n is some integer. Then, you simply notice that if n is an integer, then $2 \cdot n$ is also an integer; and hence, any element of A must also be an element of B.

3.3 Examples

Theorem 3.2 $\overline{A} \cup \overline{B} = \overline{A \cap B}$

Proof. We must show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

We will show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. Suppose $x \in \overline{A}$ then $x \notin A$. Then $x \notin A \cap B$ (if x is not in A then x is certainly not in both A and B). Hence $x \in \overline{A \cap B}$. Suppose $x \in \overline{B}$. For similar reason, $x \in \overline{A \cap B}$. Hence in either case, $x \in \overline{A \cap B}$. Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

We must show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, and so $x \notin A$ or[3] $x \notin B$. Hence either $x \in \overline{A}$ or $x \in \overline{B}$. In either case, $x \in \overline{A} \cup \overline{B}$. QED

Theorem 3.3 *$A \subseteq B$ iff $A \cap B = A$.*

Proof. We must show $A \subseteq B$ implies $A \cap B = A$ and $A \cap B = A$ implies $A \subseteq B$.

[3] Notice that $x \notin A \cap B$ **does not imply that** $x \notin A$ *and* $x \notin B$. The "and" in italics is should be an "or". Make sure you clearly understand the logic here, since this is often misunderstood by students.

Assume that $A \subseteq B$. We must show $A \cap B = A$. I.e. we must show (1) $A \cap B \subseteq A$ and (2) $A \subseteq A \cap B$. The first statement is trivial, it is always the case that $A \cap B \subseteq A$. For the second statement, assume $x \in A$. We must show $x \in A \cap B$. Since $A \subseteq B$, $x \in B$. Hence $x \in A \cap B$.

Assume $A \cap B = A$. We must show $A \subseteq B$. Let $x \in A$. Then $x \in A \cap B$ since $A = A \cap B$. Hence $x \in B$. QED

Attempt to answer these questions before looking at the answers.

Exercise 3.4 *Suppose that $f : A \to B$. Prove or disprove the following:*

1. *If $X \subseteq A$ and $Y \subseteq A$, then $f(X \cap Y) = f(X) \cap f(Y)$.*
2. *If $X \subseteq A$, $Y \subseteq A$ and f is 1-1,then $f(X \cap Y) = f(X) \cap f(Y)$.*
3. *If $X \subseteq B$ and $Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

Claim 3.5 *It not the case that if $X \subseteq A$ and $Y \subseteq A$, then $f(X \cap Y) \neq f(X) \cap f(Y)$.*

Proof of Claim 3.5. To prove this, we must find counterexample. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. And $f : A \to B$ be defined as follows: $f(1) = c$, $f(2) = b$ and $f(3) = c$. Let $X = \{1, 2\}$ and $Y = \{2, 3\}$. Then $X \cap Y = \{2\}$ and $f(X \cap Y) = f(\{2\}) = \{f(2)\} = \{b\}$. But, $f(X) \cap f(Y) = \{f(1), f(2)\} \cap \{f(2), f(3)\} = \{c, b\} \cap \{b, c\} = \{b, c\}$. Hence, $f(X \cap Y) \neq f(X) \cap f(Y)$.
QED (of Claim)

It is true that for any function $f : A \to B$ and all subsets $X, Y \subseteq A$, $f(X \cap Y) \subseteq f(X) \cap f(Y)$ (for a proof see below).

Claim 3.6 *If $X \subseteq A$, $Y \subseteq A$ and f is 1-1,then $f(X \cap Y) = f(X) \cap f(Y)$.*

Proof of Claim 3.6. Suppose that $f : A \to B$ is a 1-1 function. Let $X \subseteq A$ and $Y \subseteq A$. We must show (1) $f(X \cap Y) \subseteq f(X) \cap f(Y)$ and (2) $f(X) \cap f(Y) \subseteq f(X \cap Y)$.

Notice that (1) is true even if f is not 1-1. Let $y \in f(X \cap Y)$. Then there is an element $x \in X \cap Y$ such that $f(x) = y$. Since $x \in X \cap Y$, $x \in X$ and $x \in Y$. Therefore, $y = f(x) \in f(X)$ and $y = f(x) \in f(Y)$. Hence, $y \in f(X) \cap f(Y)$.

We now prove (2). Let $y \in f(X) \cap f(Y)$. Then $y \in f(X)$ and $y \in f(Y)$. Since $y \in f(X)$ there is $x_1 \in X$ such that $f(x_1) = y$. Since $y \in f(y)$, there is $x_2 \in Y$ such that $f(x_2) = y$. Since f is 1-1, $x_1 = x_2$. Therefore $x_1 = x_2 \in X \cap Y$; and so, $y = f(x_1) = f(x_2) \in f(X \cap Y)$. QED (of Claim)

Claim 3.7 *If $X \subseteq B$ and $Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

Proof of Claim 3.7. Let $f : A \to B$ be any function and suppose $X \subseteq B$ and $Y \subseteq B$. We must show $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$ and $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$.

Suppose $x \in f^{-1}(X \cap Y)$. Then $f(x) \in X \cap Y$. Hence $f(x) \in X$ and $f(x) \in Y$. Since $f(x) \in X$, $x \in f^{-1}(X)$. Since $f(x) \in Y$, $x \in f^{-1}(Y)$. Therefore $x \in f^{-1}(X) \cap f^{-1}(Y)$.

Suppose $x \in f^{-1}(X) \cap f^{-1}(Y)$. Then $x \in f^{-1}(X)$ and $x \in f^{-1}(Y)$. Therefore, $f(x) \in X$ and $f(x) \in Y$. Hence, $f(x) \in X \cap Y$; and so, $x \in f^{-1}(X \cap Y)$. QED (of Claim)