Logical and Probabilistic Models of Belief Change

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Plan

✓ Introduction to belief revision, AGM

Day 2 Possible worlds models, Bayesian models (continued), Updating probabilities (time permitting)

Day 3 Updating probabilities, Lottery Paradox, Preface Paradox, Review Paradox, Iterated belief revision, Context shifts, Becoming aware

Day 4 The value of learning, Lottery Paradox, Preface Paradox, Review Paradox, Iterated belief revision, Context shifts, Becoming aware (continued)

Day 5 Interactive epistemology (Agreement Theorems), Convergence Theorems

Evaluating counterexamples

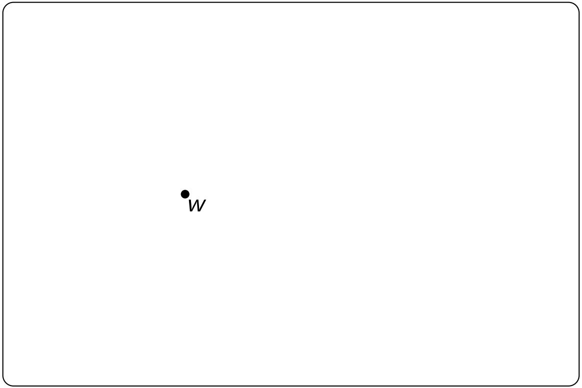
". . . information about how I learn some of the things I learn, about the sources of my information, or about what I believe about what I believe and don't believe. If the story we tell in an example makes certain information about any of these things relevant, then it needs to be included in a proper model of the story, if it is to play the right role in the evaluation of the abstract principles of the model."

Robert Stalnaker. *Iterated Belief Revision*. Erkenntnis 70, pp. 189 - 209, 2009.

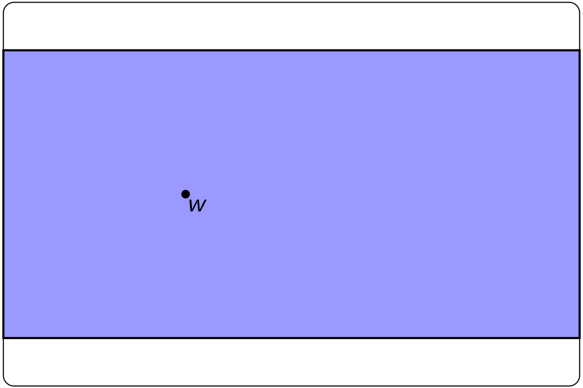
Belief Revision: The Semantic View

A. Grove. *Two modelings for theory change*. Journal of Philosophical Logic, 17, pgs. 157 - 170, 1988.

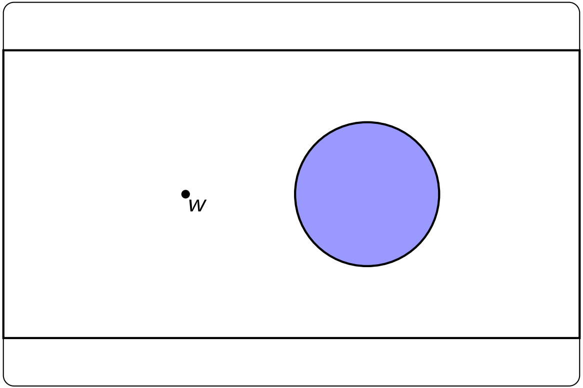
EP. *Dynamic Epistemic Logic II: Logics of information change*. Philosophy Compass, Vol. 8, Iss. 9, pgs. 815 - 833, 2013.



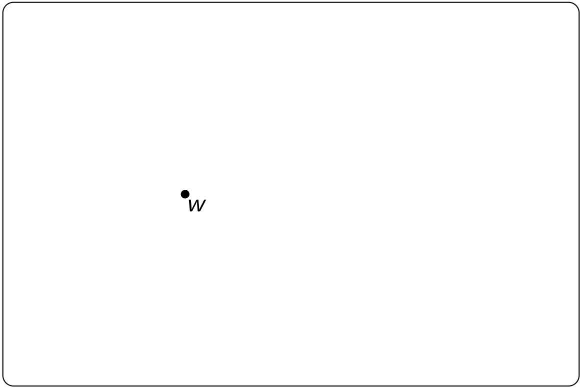
- The set of states, with a distinguished state denoting the "actual world"



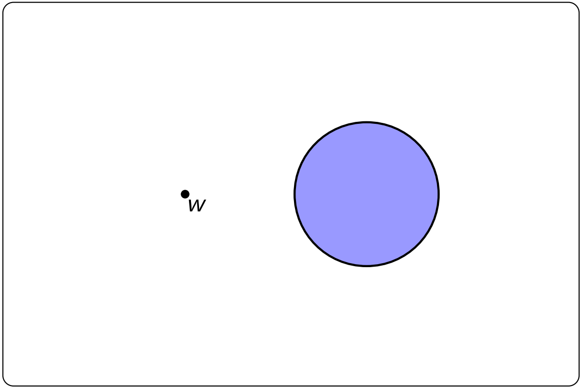
- The set of states, with a distinguished state denoting the "actual world".
- The agent's (hard) information (i.e., the states consistent with what the agent knows)



- The agent's (hard) information (i.e., the states consistent with what the agent knows)
- The agent's beliefs (soft information—the states consistent with what the agent believes)



- The states consistent with what the agent knows with a distinguished state (the "actual world")
- Each state is associated with a propositional valuation for the underlying propositional language



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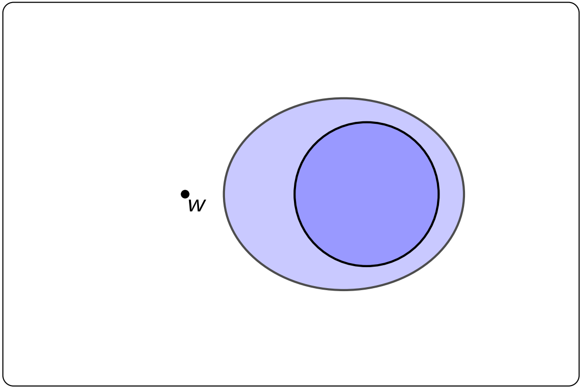

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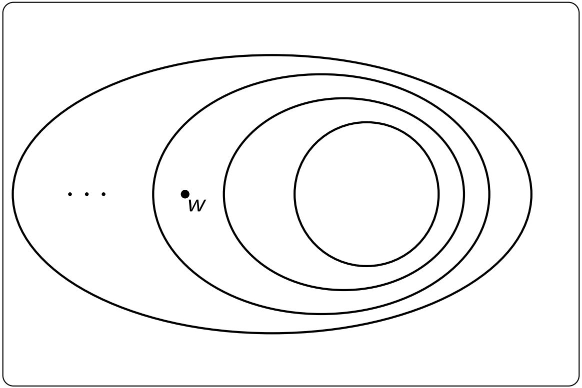
- The agent's "contingency plan"



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Sphere Models

Let W be a set of states, A set $\mathcal{F} \subseteq \wp(W)$ is called a system of spheres provided:

- For each $S, S' \in \mathcal{F}$, either $S \subseteq S'$ or $S' \subseteq S$
- For any $P \subseteq W$ there is a smallest $S \in \mathcal{F}$ (according to the subset relation) such that $P \cap S \neq \emptyset$
- The spheres are non-empty $\bigcap \mathcal{F} \neq \emptyset$ and cover the entire information cell $\bigcup \mathcal{F} = W$

Let $\mathcal{F}$ be a system of spheres on W: for $w, v \in W$, let

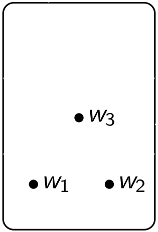
$$w \preceq_{\mathcal{F}} v \text{ iff for all } S \in \mathcal{F}, \text{ if } v \in S \text{ then } w \in S$$

Then, $\preceq_{\mathcal{F}}$ is reflexive, transitive, and well-founded.

$w \preceq_{\mathcal{F}} v$ means that no matter what the agent learns in the future, as long as world v is still consistent with his beliefs and w is still epistemically possible, then w is also consistent with his beliefs.

Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$



Belief Revision via Plausibility

- $W = \{w_1, w_2, w_3\}$
- $w_1 \preceq w_2$ and $w_2 \preceq w_1$ (w_1 and w_2 are equi-plausbile)
- $w_1 \prec w_3$ ($w_1 \preceq w_3$ and $w_3 \not\preceq w_1$)
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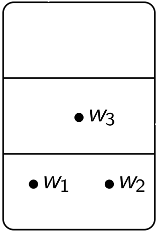

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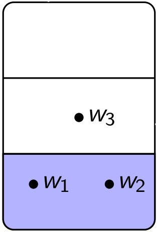
- $\{w_1, w_2\} \subseteq Min_{\preceq}([w_i])$



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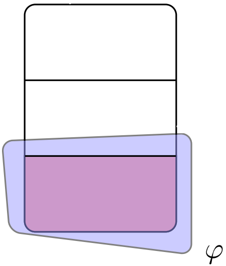
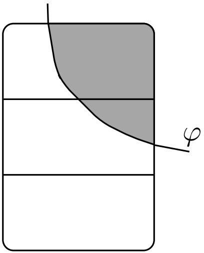


Belief: $B\varphi$

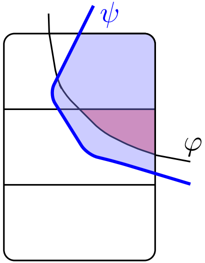
$$Min_{\preceq}(W) \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

Belief Revision via Plausibility



Conditional Belief: $B^{\varphi}\psi$

Belief Revision via Plausibility



Conditional Belief: $B^{\varphi}\psi$

$$Min_{\preceq}(\llbracket\varphi\rrbracket_{\mathcal{M}}) \subseteq \llbracket\psi\rrbracket_{\mathcal{M}}$$

Plausibility Models

Epistemic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \text{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w \sim_i v$, then $\mathcal{M}, v \models \varphi$

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Epistemic-Plausibility Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

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Plausibility Relation: $\preceq_i \subseteq W \times W$ where $w \preceq_i v$ means "w is at least as plausible as v."

Assumptions:

1. $\preceq_i$ is reflexive and transitive (and well-founded)
2. *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.
3. *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

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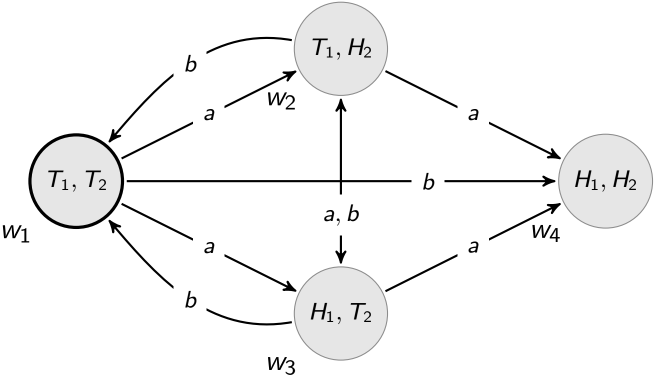
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w{\sim_i}v$, then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i\varphi$ if for each $v \in Min_{\preceq_i}([w]_i)$, $\mathcal{M}, v \models \varphi$
  $[w]_i = \{v \mid w \sim_i v\}$ is the agent's **information cell**.

Example



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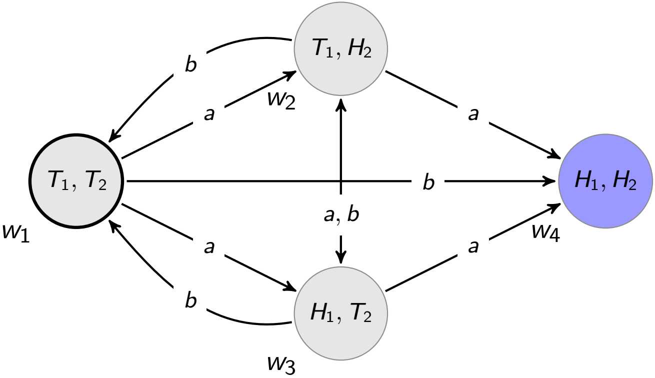

- $w_1 \models B_a(H_1 \wedge H_2) \wedge B_b(H_1 \wedge H_2)$

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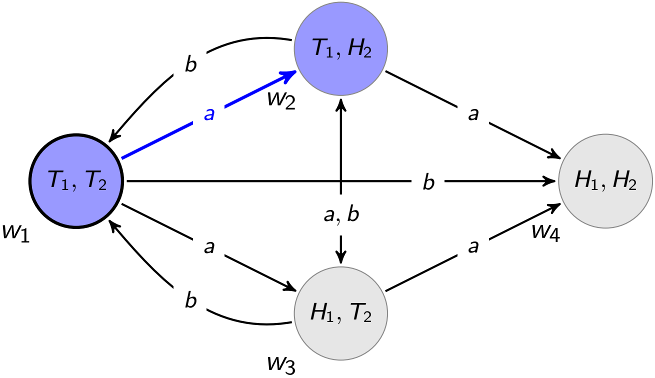

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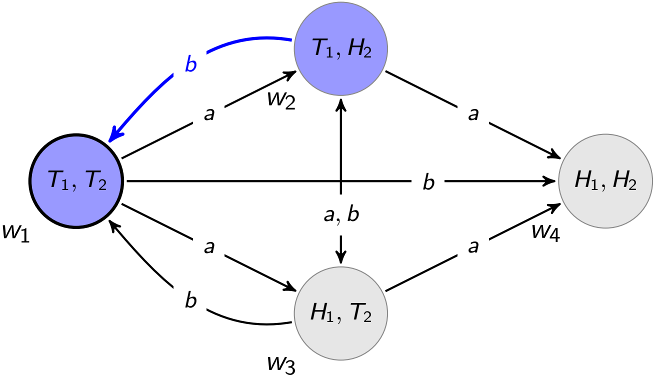

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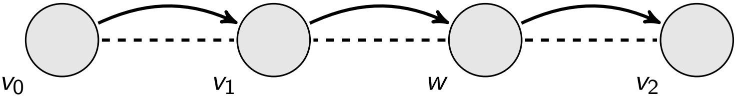
Grades of Doxastic Strength



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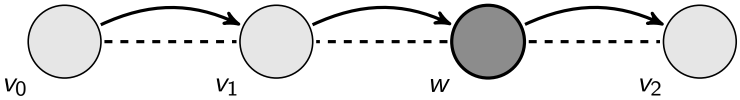

Suppose that w is the current state.

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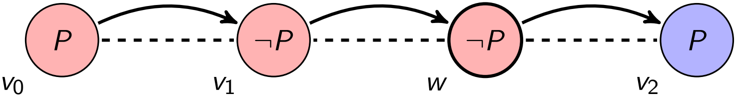

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- **Belief** (BP)

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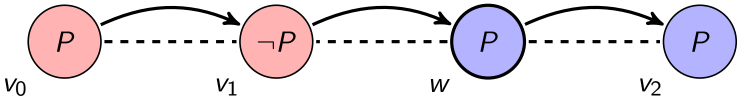

Suppose that w is the current state.

- **Belief** (BP)
- **Robust Belief** ($[\preceq]P$)

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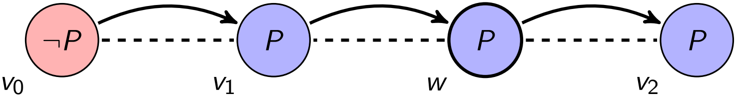

Suppose that w is the current state.

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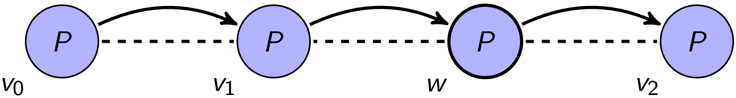

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- **Belief** (BP)
- **Robust Belief** ($[\preceq]P$)
- **Strong Belief** ($B^s P$)
- **Knowledge** (KP)

Is $B\varphi \to B^{\psi}\varphi$ valid?

Is $B\varphi \to B^{\psi}\varphi$ valid?

Is $B^{\alpha}\varphi \to B^{\alpha \wedge \beta}\varphi$ valid?

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid?

Is $B^{\alpha}\varphi \rightarrow B^{\alpha\wedge\beta}\varphi$ valid?

Is $B\varphi \rightarrow B^{\psi}\varphi \vee B^{\neg\psi}\varphi$ valid?

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Is $B\varphi \to B^{\psi}\varphi \vee B^{\neg\psi}\varphi$ valid?

Exercise: Prove that B, B^{φ} and B^{s} are definable in the language with K and $[\preceq]$ modalities.

$\mathcal{M}, w \models B^{\varphi}\psi$ if for each $v \in Min_{\preceq}([w] \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$ and $[w] = \{v \mid w \sim v\}$

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Core Logical Principles:

1. $B^{\varphi}\varphi$
2. $B^{\varphi}\psi \rightarrow B^{\varphi}(\psi \vee \chi)$
3. $(B^{\varphi}\psi_1 \wedge B^{\varphi}\psi_2) \rightarrow B^{\varphi}(\psi_1 \wedge \psi_2)$
4. $(B^{\varphi_1}\psi \wedge B^{\varphi_2}\psi) \rightarrow B^{\varphi_1 \vee \varphi_2}\psi$
5. $(B^{\varphi}\psi \wedge B^{\psi}\varphi) \rightarrow (B^{\varphi}\chi \leftrightarrow B^{\psi}\chi)$

J. Burgess. *Quick completeness proofs for some logics of conditionals*. Notre Dame Journal of Formal Logic 22, 76 – 84, 1981.

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
  i knows φ iff i continues to believe φ given any new information

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- $\mathcal{M}, w \models B_i^{s}\varphi$ iff $\mathcal{M}, w \models B_i\varphi$ and $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \neg K_i(\psi \rightarrow \neg\varphi)$.
  i strongly believes φ iff *i* believes φ and continues to believe φ given any evidence (truthful or not) that is not known to contradict φ.

Dynamic Epistemic Logic

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In the simplest case, we model an agent's acquisition of knowledge by the elimination of possibilities from an initial epistemic model.

Finding out that φ

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$$

$$\Big\Downarrow$$

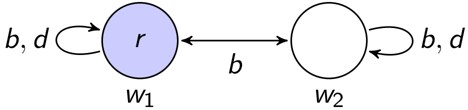
Find out that φ

$$\Big\Downarrow$$

$$\mathcal{M}' = \langle W', \{\sim_i'\}_{i\in\mathcal{A}}, \{\preceq_i'\}_{i\in\mathcal{A}}, V|_{W'}\rangle$$

Example: College Park and Amsterdam

Recall the College Park agent who doesn't know whether it's raining in Amsterdam, whose epistemic state is represented by the model:



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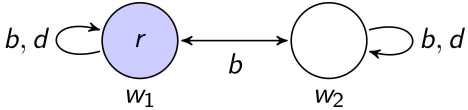

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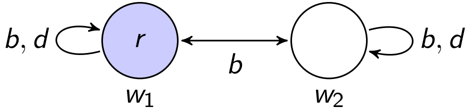

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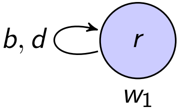

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Model Update

We can easily give a formal definition that captures the idea of knowledge acquisition as the elimination of possibilities.

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In the single-agent case, this models the agent learning φ. In the multi-agent case, this models all agents *publicly* learning φ.

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The language of Public Announcement Logic (PAL) is given by:

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Read $\langle !\varphi\rangle\psi := \neg[!\varphi]\neg\psi$ as "after a true announcement of φ, ψ."

Public Announcement Logic

The truth clause for the dynamic operator $[!\varphi]$ is:

- $\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ *implies* $\mathcal{M}_{|\varphi}, w \models \psi$.

  So if φ is false, $[!\varphi]\psi$ is vacuously true. Here is the $\langle !\varphi \rangle$ clause:

- $\mathcal{M}, w \models \langle !\varphi \rangle\psi$ iff $\mathcal{M}, w \models \varphi$ *and* $\mathcal{M}_{|\varphi}, w \models \psi$.

Main Idea: we evaluate $[!\varphi]\psi$ and $\langle !\varphi \rangle\psi$ not by looking at *other worlds in the same model*, but rather by looking at a new model.

Public Announcement Logic

Suppose $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$ is a multi-agent Kripke Model

$$\mathcal{M}, w \models [!\psi]\varphi \text{ iff } \mathcal{M}, w \models \psi \text{ implies } \mathcal{M}|_\psi, w \models \varphi$$

where $\mathcal{M}|_\psi = \langle W', \{\sim_i'\}_{i\in\mathcal{A}}, \{\preceq_i'\}_{i\in\mathcal{A}}, V'\rangle$ with

- $W' = W \cap \{w \mid \mathcal{M}, w \models \psi\}$
- For each i, $\sim_i' = \sim_i \cap (W' \times W')$
- For each i, $\preceq_i' = \preceq_i \cap (W' \times W')$
- for all $p \in \text{At}$, $V'(p) = V(p) \cap W'$

Public Announcement Logic

$$[!\psi]p \quad \leftrightarrow \quad (\psi \to p)$$

Public Announcement Logic

$$
\begin{aligned}
[!\psi]p &\quad\leftrightarrow\quad (\psi \rightarrow p) \\
[!\psi]\neg\varphi &\quad\leftrightarrow\quad (\psi \rightarrow \neg[!\psi]\varphi)
\end{aligned}
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Public Announcement Logic

$$\begin{aligned} [!\psi]p &\leftrightarrow (\psi \to p) \\ [!\psi]\neg\varphi &\leftrightarrow (\psi \to \neg[!\psi]\varphi) \\ [!\psi](\varphi \wedge \chi) &\leftrightarrow ([!\psi]\varphi \wedge [!\psi]\chi) \end{aligned}$$

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Theorem Every formula of Public Announcement Logic is equivalent to a formula of Epistemic Logic.

Public Announcement vs. Conditional Belief

Are $[!\varphi]B\psi$ and $B^{\varphi}\psi$ different?

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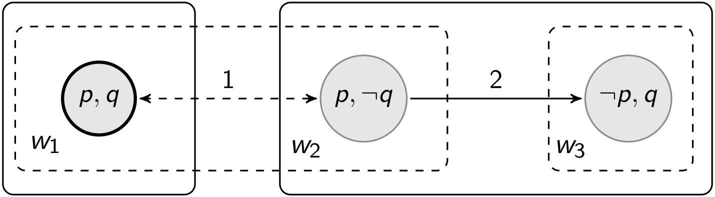

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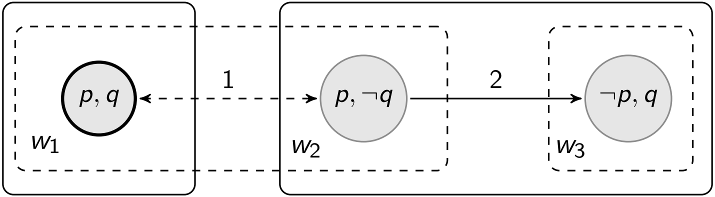

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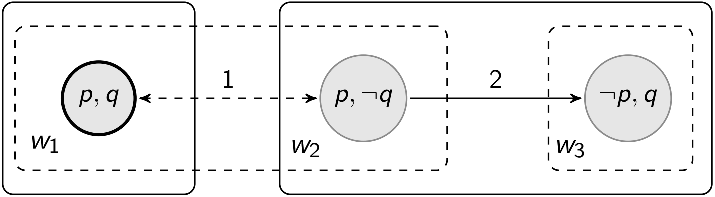

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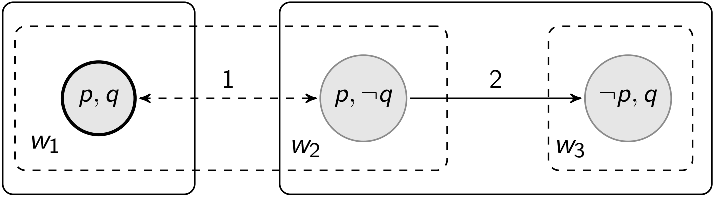

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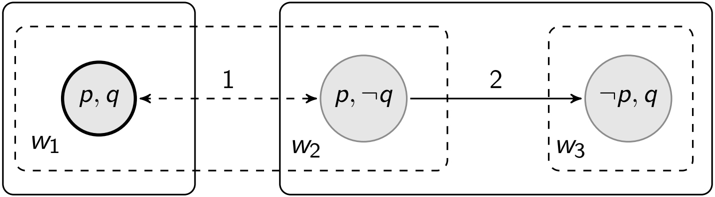

- $w_1 \models B_1 B_2 q$
- $w_1 \models B_1^p B_2 q$
- $w_1 \models [!p]\neg B_1 B_2 q$
- More generally, $B_i^p(p \wedge \neg K_i p)$ is satisfiable but $[!p]B_i(p \wedge \neg K_i p)$ is not.

The Logic of Public Observation

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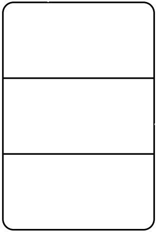
Belief Revision via Plausibility



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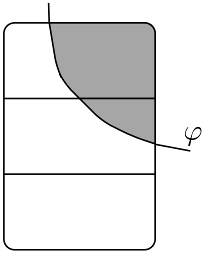

Incorporate the new information φ

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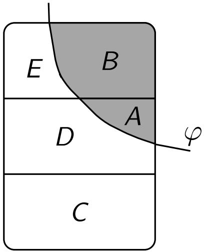

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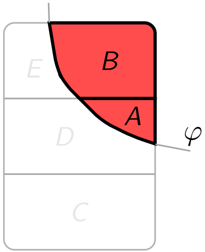

Public Announcement: Information from an infallible source
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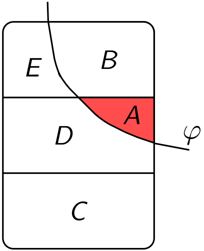

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Conservative Upgrade: Information from a trusted source
$(\uparrow\varphi)$: $A \prec_i C \prec_i D \prec_i B \cup E$

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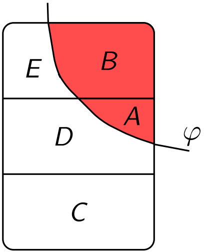

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Radical Upgrade: Information from a strongly trusted source
$(\Uparrow\varphi)$: $A \prec_i B \prec_i C \prec_i D \prec_i E$

$$K_0 \qquad \Longrightarrow \qquad K_t$$

$$K_0 \overset{\text{Find out that } \varphi}{\Longrightarrow} K_t = K_0 * \varphi$$

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$$\mathcal{M}_0 \overset{\text{Find out that } \varphi}{\Longrightarrow} \mathcal{M}_t = \mathcal{M}_0^{\Uparrow\varphi}$$

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Find out that φ

$$K_0 \quad \Longrightarrow \quad K_t = K_0 * \varphi$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

Find out that φ

$$\mathcal{M}_0 \quad \Longrightarrow \quad \mathcal{M}_t = \mathcal{M}_0^{\Uparrow\varphi}$$

$$= \mathcal{M}_0^{\uparrow\varphi}$$

Bayesian Models

Conceptions of Belief

Binary: "all-out" belief. For any statement p, the agent either does or does not believe p. It is natural to take an *unqualified* assertion as a statement of belief of the speaker.

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Eric Schwitzgebel. *Belief*. In The Stanford Encyclopedia of Philosophy.

Franz Huber. *Formal Theories of Belief*. In The Stanford Encyclopedia of Philosophy.

Conceptions of Beliefs: Questions

What are the *formal constraints* on rational belief?

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What are the *formal constraints* on rational belief?

- rational graded beliefs should obey the laws of probability
- rational all-out beliefs should be consistent/deductively closed
- how should we justify these constraints?

D. Christensen. *Putting Logic in its Place*. Oxford University Press.

Suppose that W is a set of states (the *set of outcomes*).

A σ-algebra is a set $\Sigma \subseteq \wp(W)$ such that

- $W \in \Sigma$
- If $A \in \Sigma$, then $\overline{A} \in \Sigma$
- If $\{A_i\}$ is a countable collection of sets from Σ, then $\bigcup_i A_i \in \Sigma$

A **probability function** is a function $p : \Sigma \to [0, 1]$ satisfying:

- $p(W) = 1$
- $p(A \cup B) = p(A) + p(B)$ whenever $A \cap B = \emptyset$

(W, Σ, p) is called a probability space.

Probability

Kolmogorov Axioms:

1. For each E, $0 \leq p(E) \leq 1$
2. $p(W) = 1$, $p(\emptyset) = 0$
3. If $E_1, \ldots, E_n, \ldots$ are pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$), then $p(\bigcup_i E_i) = \sum_i p(E_i)$

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- $p(\overline{E}) = 1 - p(E)$ ($\overline{E}$ is the complement of E)
- If $E \subseteq F$ then $p(E) \leq p(F)$
- $p(E \cup F) = p(E) + p(F) - p(E \cap F)$

Suppose that $(\mathcal{L}, \models)$ is a logic. A probability function is a map $p : \mathcal{L} \rightarrow [0, 1]$ such that

1. For each E, $0 \leq p(\varphi) \leq 1$
2. $p(\varphi) = 1$ if $\models \varphi$
3. If $p(\varphi \vee \psi) = p(\varphi) + p(\psi)$ when $\models \neg(\varphi \wedge \psi)$.

I.J. Good. *46,656 Varieties of Bayesians*. Good Thinking: The Foundations of Probability and Its Applications, University of Minnesota Press (1983).

Conditional Probability

The probability of E *given* F, dented $p(E|F)$, is defined to be

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

provided $P(F) > 0$.

Bayes Theorem

$$p(H|E) = p(E|H)\frac{p(H)}{p(E)}$$

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$$p(H|E) = p(E|H)\frac{p(H)}{p(E)}$$

Bayes theorem is important because it expresses the quantity $p(H|E)$ (the probability of a hypothesis H given the evidence E) —which is something people often find hard to assess—in terms of quantities that can be drawn directly from experiential knowledge.

Example: Suppose you are in a casino and you hear a person at the next gambling table announce "Twelve". We want to know wether he was rolling a pair of dice or a roulette wheel.

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Based on our background knowledge of gambling we have $p(Twelve \mid Dice) = 1/36$ and $p(Twelve \mid Roulette) = 1/38$.

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Based on our observations about the casino, we can judge the prior probabilities $p(\mathit{Dice})$ and $p(\mathit{Roulette})$.

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But this is now enough to *calculate* the required probabilities.

Extensions and variations

- Dempster-Shafer belief functions: $Bel : A \to [0, 1]$ are *super-additive*, $Bel(A) + Bel(B) \leq Bel(A \cup B)$ if $A \cap B = \emptyset$. The the number $Bel(A)$ represents the strength with which A is supported by the agent's knowledge or belief base.

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- Non-standard probability: $\mu : \Sigma \to \mathbb{R}^*$

- Halpern Plausibility Functions: $\mu : \Sigma \to (D, \preceq)$.

Imprecise Probabilities

- 1. What is the probability that a fair coin will land hands?
  2. What is the probability of a coin of unknown bias will land heads?

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- Ellsberg Paradox

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	30	60	
Lotteries	Blue	Yellow	Green
L_1	$1M$	0	0
L_2	0	$1M$	0
L_3	$1M$	0	$1M$
L_4	0	$1M$	$1M$

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$$L_1 \succeq L_2 \text{ iff } L_3 \succeq L_4$$

Indeterminate Probability

- Allow probability functions to take on sets of values instead of a single value
- Work with sets of probabilities rather than a single probability

Dutch Book Arguments

Should a rational agent's graded beliefs satisfy the laws of probability?

Ramsey, de Finetti and Savage (1)

How do we *measure* a (rational) agent's subjective probabilities?

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What we need: systematic procedures for linking the probability calculus (graded beliefs) to claims about objectively observable behavior, such as preferences revealed by choice behavior.

Ramsey, de Finetti and Savage (2)

Suppose we are wondering about Ann's degree of belief about whether a coin will land heads (H) or tails (T).

Offer Ann two bets:

- L_1 If the coin lands heads, you win a sports car; otherwise you win nothing
- L_2 If the coin does not land heads, you win a sports car; otherwise you win nothing.

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Offer Ann two bets:

- L_1 If the coin lands heads, you win a sports car; otherwise you win nothing
- L_2 If the coin does not land heads, you win a sports car; otherwise you win nothing.

If Ann chooses L_1, she believes H is more probable than T
If Ann chooses L_2, she believes T is more probable than H
If Ann is indifferent, she believes H and T are equally probable (i.e., $p_A(H) = p_A(T) = 1/2$)

Accuracy Arguments

J. Joyce. *A nonpragmatic vindication of probabilism*. Philosophy of Science 65, 575603 (1998).

R. Pettigrew. *Epistemic Utility Arguments for Probabilism*. Stanford Encyclopedia of Philosophy, 2015.

R. Pettigrew. *Accuracy and the Laws of Credence*. Oxford University Press, 2016.

Accuracy

Accuracy. An epistemic agent ought to approximate the truth. In other words, she ought to minimize her inaccuracy.

Accuracy

Accuracy (Synchronic expected local). An agent ought to minimize the **expected local inaccuracy** of her degrees of credence in all propositions $A \subseteq W$ by the lights of her current belief function, relative to a **legitimate local inaccuracy measure** and over the set of worlds that are currently epistemically possible for her.

Accuracy (Synchronic expected global). An agent ought to minimize the **expected global inaccuracy** of her current belief function by the lights of her current belief function, relative to a legitimate global inaccuracy measure and over the set of worlds that are currently epistemically possible for her

Measuring Inaccuracy

Alethic Vindication The ideal credence function at world w is the omniscient credence function at w, namely, v_w.

Perfectionism The accuracy of a credence function at a world is its proximity to the ideal credence function at that world.

Squared Euclidean Distance Distance between credence functions is measured by squared Euclidean distance.

B. de Finetti. *Theory of Probability*. John Wiley and Sons, 1974.

J. Pred, R. Seiringer, E. Lieb, D. Osherson, H. V. Poor, and S. Kulkarni. *Probabilistic Coherence and Proper Scoring Rules*. IEEE Transactions on Information Theory, 2009.

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de Finetti, Predd et al., Lindley, . . . : The two defects are equivalent.

Brier Score

$$\mathcal{E} = (E, F) \text{ with } E \subseteq F$$
$$\mathbf{f} = (0.6, 0.9)$$

Brier Score

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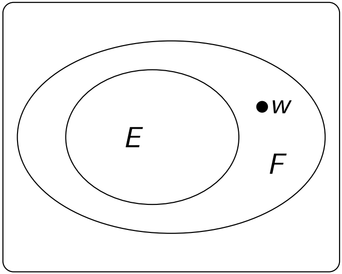

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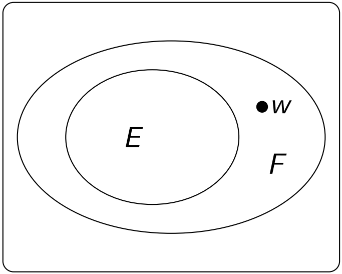

Penalty: $(0 - 0.6)^2 + (1 - 0.9)^2 = 0.37$

Proper Scoring Rule

$\mathcal{E} = (E, F)$ with $E \subseteq F$

$\mathbf{f} = (0.6, 0.9)$

Expected Penalty for E:
$0.6 * (1 - 0.6)^2 + 0.4 * (0 - 0.6)^2 = 0.230$

Expected Penalty for E **by lying**:
$0.6 * (1 - 0.65)^2 + 0.4 * (0 - 0.65)^2 = 0.2425$

Proper Scoring Rule

Suppose your probability for an event E is p, that your announced probability is x, and that your penalty assessed according ot the rule $(1 - x)^2$ if E comes out true; $(0 - x)^2$ otherwise. Then your expected penalty is uniquely minimized by choosing $x = p$.

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Absolute Deviation

Expected Penalty for E:
$0.6 * |1 - 0.6| + 0.4 * |0 - 0.6| = 0.48$

Expected Penalty for E **by lying**:
$0.6 * |1 - 0.65| + 0.4 * |0 - 0.65| = 0.47$

$\mathcal{E} = (E, F)$ with $E \subseteq F$

$\mathbf{f} = (0.6, 0.9)$
$\mathbf{f}' = (0.95, 0.55)$

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Penalties:

Possibility	$\mathbf{f}$	$\mathbf{f}'$
E true, F true	0.17	0.205
E false, F true	0.37	1.105
E false, F false	1.17	1.205

S is a sample space. Subsets of S are events. Let $\mathcal{E} = (E_1, \ldots, E_n)$ be a vector of events.

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A function $s : \{0,1\} \times [0,1] \to [0,\infty]$ is a **proper scoring rule** in case:

1. $ps(1,x) + (1-p)s(0,x)$ is uniquely minimized at $x = p$ for $p \in [0,1]$.
2. s is continuous. For $i \in \{0,1\}$, $\lim_{n\to\infty} s(i,x_n) = s(i,x)$ for any sequence x_n from $[0,1]$ converging to x.

Penalty

Given a proper scoring rule s, the penalty P_s based on s for forecast $\mathbf{f}$ and $w \in S$ is given by:

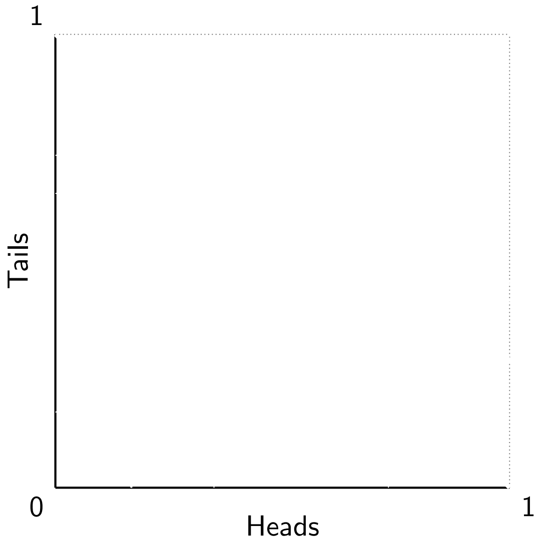
$$P_s(w, \mathbf{f}) = \sum_{i=1}^{n} s(\chi_{E_i}(w), f_i)$$

$\mathbf{f}$ is **weakly dominated** by $\mathbf{g}$ in case $P_s(w, \mathbf{g}) \leq P_s(w, \mathbf{f})$ for all $w \in S$.

$\mathbf{f}$ is **strongly dominated** by $\mathbf{g}$ in case $P_s(w, \mathbf{g}) < P_s(w, \mathbf{f})$ for all $w \in S$.

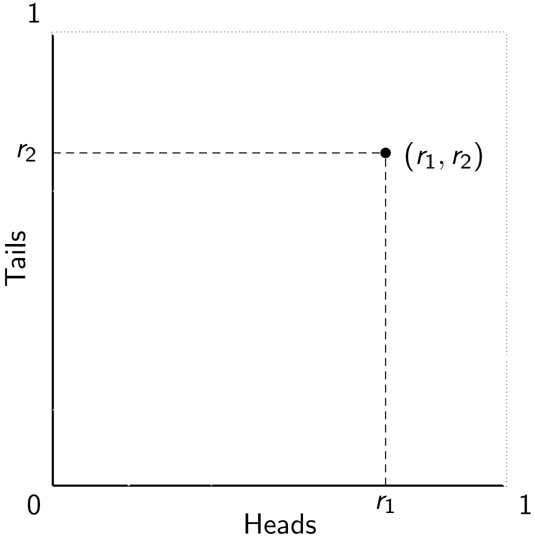
Theorem Let $\mathbf{f}$ be a forecast.

1. If $\mathbf{f}$ is coherent, then it is not weakly dominated by any forecast $\mathbf{g} \neq \mathbf{f}$
2. If $\mathbf{f}$ is incoherent, then it is strongly dominated by some coherent forecast $\mathbf{g}$



w_H: The coin is facing heads up.

w_T: The coin is facing tails up.



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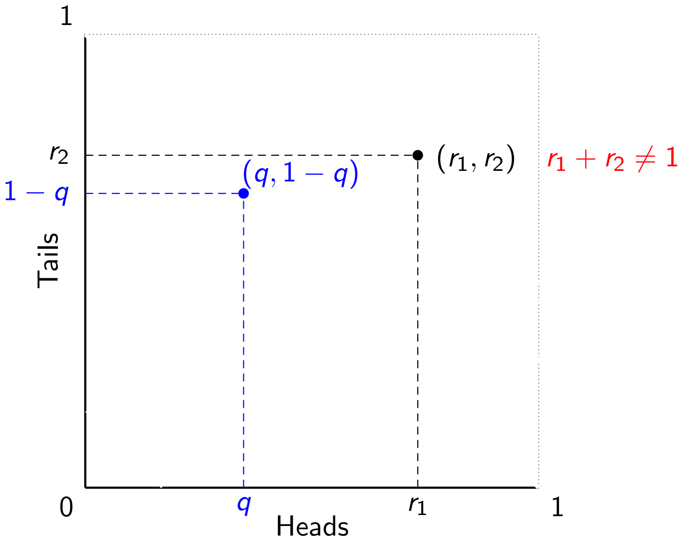

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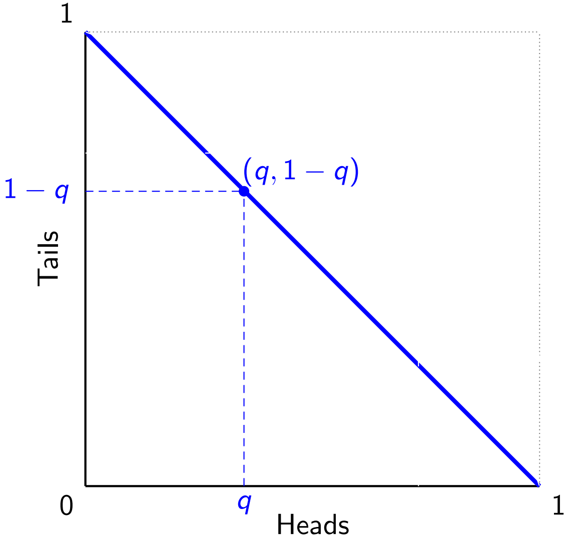

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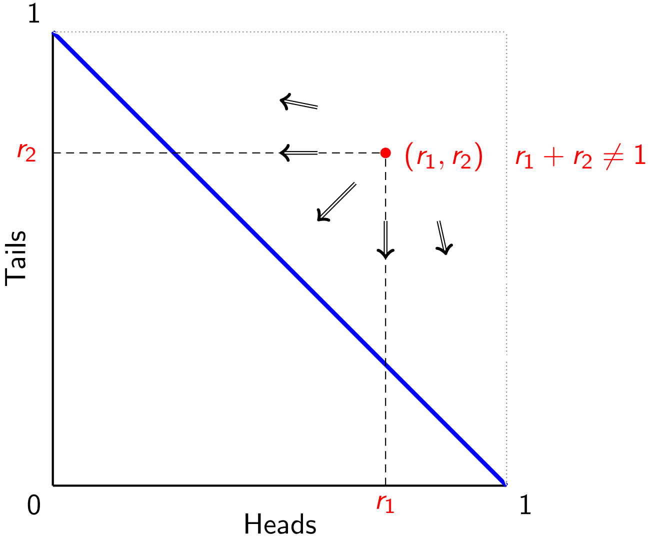

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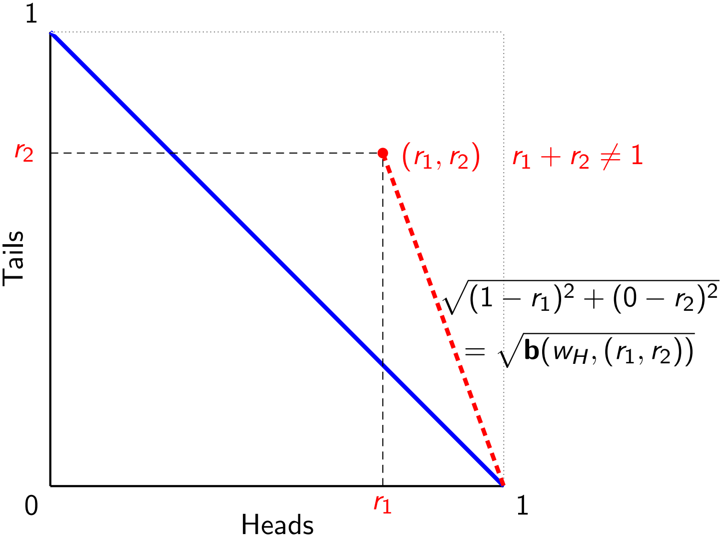

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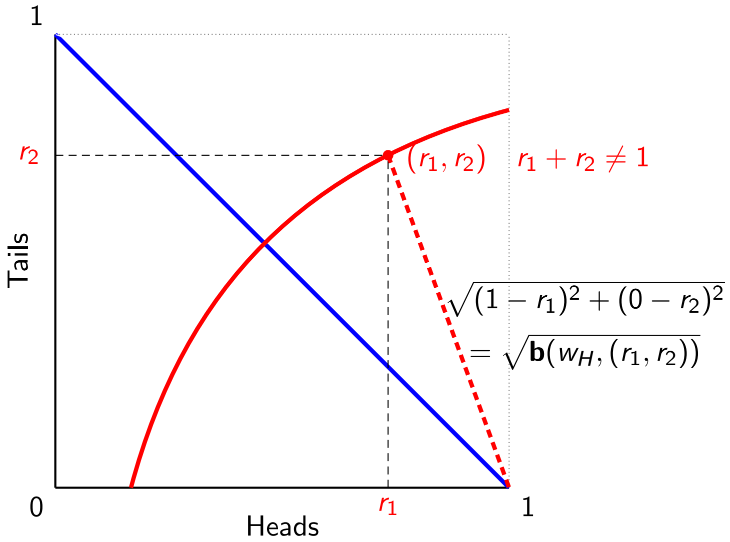

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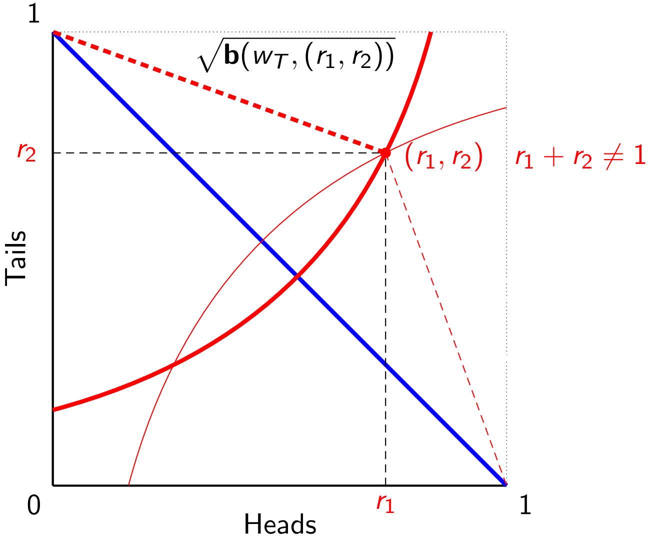

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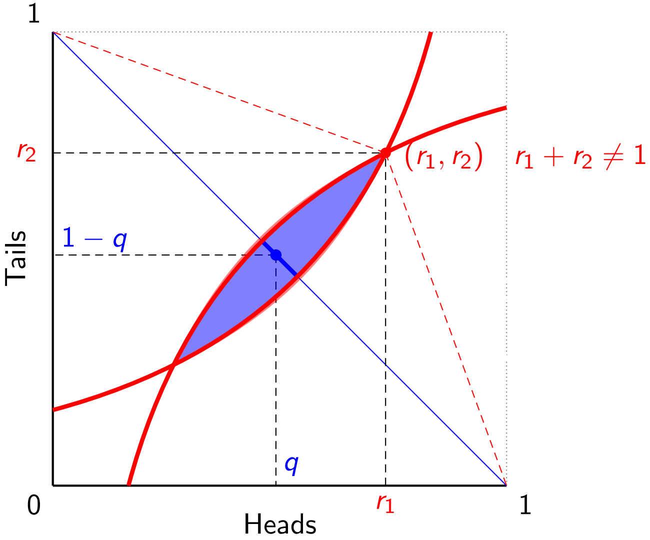

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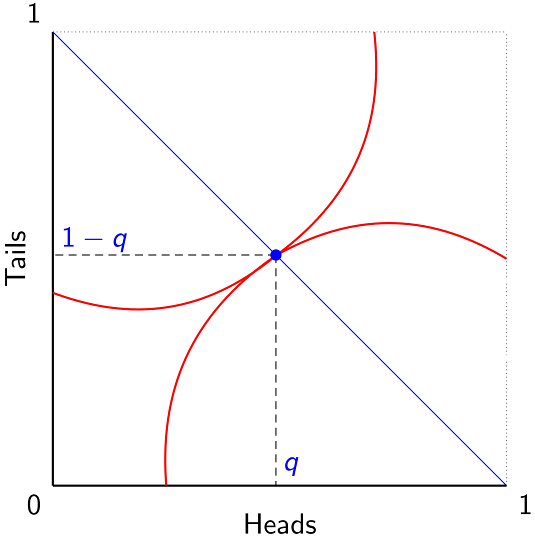

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$$K_0 \qquad \Longrightarrow \qquad K_t$$

Learn that φ

Suppose that φ

$$K_0 \qquad \Longrightarrow \qquad K_t = K_0 * \varphi$$

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$$
\begin{array}{ccc}
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K_0 & \Longrightarrow & K_t = K_0 * \varphi \\
\updownarrow & & \updownarrow \\
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$$