

# Reasoning in Games: Players as Programs

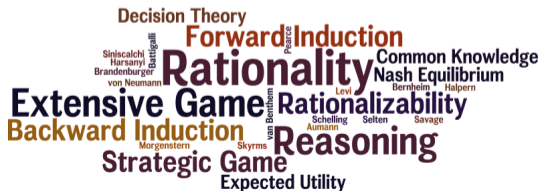
## Lecture 1

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[pacuit.org](http://pacuit.org)



# Plan

**Monday** Epistemic utility theory, Decision- and game-theoretic background: Nash equilibrium

**Tuesday** Introduction to game theory: rationalizability, epistemic game theory, forward and backward induction; Iterated games and learning, Skyrms's model of rational deliberation I

**Wednesday** Skyrms's model of rational deliberation II; Introduction to webppl; Game-theoretic reasoning in webppl

**Thursday** Coordination games (comparing Skyrms's model of deliberation and the webppl approach)

**Friday** Models of game-theoretic reasoning

# The Guessing Game



Guess a number between 1 & 100.  
The closest to  $\frac{2}{3}$  of the average wins.

[pacuit.org/games/avg](http://pacuit.org/games/avg)

## The Guessing Game, again



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What number should you guess?

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What number should you guess? 100



# The Guessing Game



Guess a number between 1 & 100.  
The closest to  $2/3$  of the average wins.

What number should you guess? ~~100~~, 99

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Guess a number between 1 & 100.  
The closest to  $2/3$  of the average wins.

What number should you guess? ~~100~~, ~~99~~, ..., 67

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Guess a number between 1 & 100.  
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What number should you guess? ~~100~~, ~~99~~, ..., ~~67~~, ..., 2, 1

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Guess a number between 1 & 100.  
The closest to  $2/3$  of the average wins.

What number should you guess? ~~100~~, ~~99~~, ..., ~~67~~, ..., ~~2~~, **1**

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Suppose that you are randomly paired with another person from class. What number would you write down?

[pacuit.org/games/td](http://pacuit.org/games/td)

# Plan

- ▶ Expected utility
- ▶ Basic game-theoretic reasoning: Nash equilibrium, rationalizability
- ▶ Epistemic game theory, correlated equilibrium
- ▶ Backward and forward induction

# Decision Problems

In many circumstances the decision maker doesn't get to choose outcomes directly, but rather chooses an instrument that affects what outcome actually occurs.

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Economists distinguish between choice under:

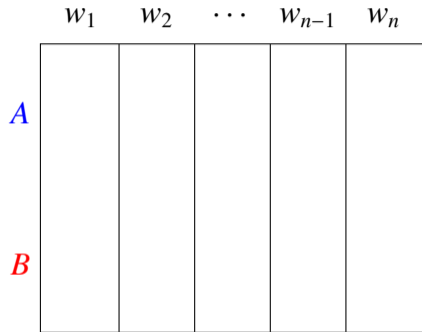
- ▶ *certainty*: highly confident about the relationship between actions and outcomes
- ▶ *risk*: clear sense of possibilities and their likelihoods
- ▶ *uncertainty*: the relationship between actions and outcomes is so imprecise that it is not possible to assign likelihoods

# Decision Problems

*A*

*B*

# Decision Problems

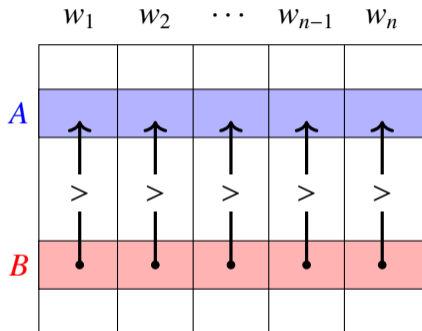


# Decision Problems

	$w_1$	$w_2$	$\dots$	$w_{n-1}$	$w_n$
$A$					
$B$					

An **act** is a function  $F : W \rightarrow O$

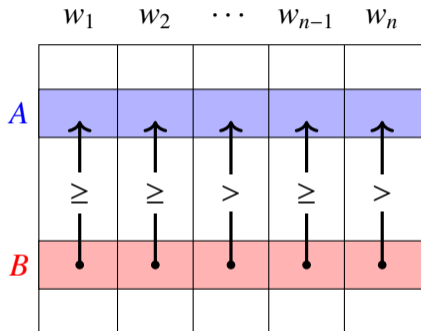
# Strict Dominance



$$\forall w \in W, u(A(w)) > u(B(w))$$

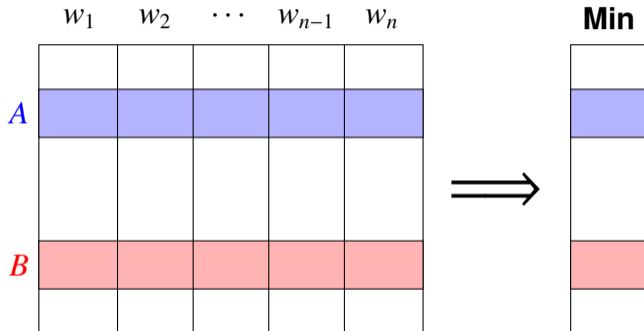


# Weak Dominance



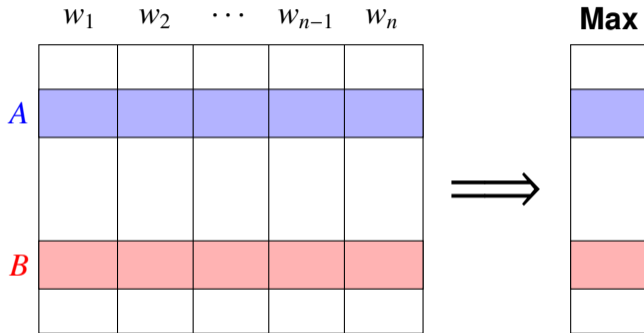
$$\forall w \in W, u(A(w)) \geq u(B(w)) \text{ and } \exists w \in W, u(A(w)) > u(B(w))$$

# MaxMin (Security)



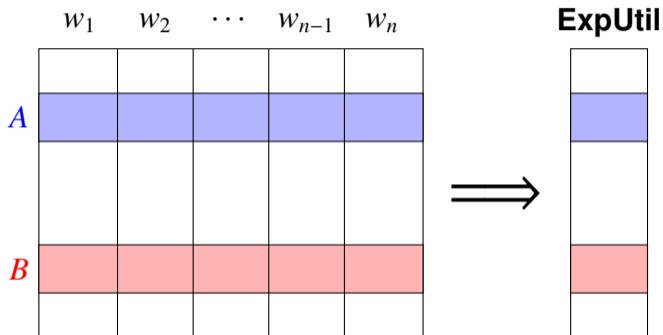
$$\min(\{u(A(w)) \mid w \in W\}) > \min(\{u(B(w)) \mid w \in W\})$$

# MaxMax



$$\max(\{u(A(w)) \mid w \in W\}) > \max(\{u(B(w)) \mid w \in W\})$$

# Maximize (Subjective) Expected Utility



$$\sum_{w \in W} P_A(w) * u(A(w)) > \sum_{w \in W} P_A(w) * u(B(w))$$

# Subjective Expected Utility

**Probability:** Suppose that  $W = \{w_1, \dots, w_n\}$  is a finite set of states. A probability function on  $W$  is a function  $P : W \rightarrow [0, 1]$  where  $\sum_{w \in W} P(w) = 1$  (i.e.,  $P(w_1) + P(w_2) + \dots + P(w_n) = 1$ ).

Suppose that  $A$  is an act for a set of outcomes  $O$  (i.e.,  $A : W \rightarrow O$ ). The **expected utility** of  $A$  is:

$$\sum_{w \in W} P(w) * u(A(w))$$

$$EU(A) = \sum_{o \in O} P_A(o) \times U(o)$$

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Probability of outcome  $o$  conditional on  $A$

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Classical:  $P_A(o) = \sum_{s \in \mathcal{S}} P(s) f_{A,s}(o)$ , where

$$f_{A,s}(o) = \begin{cases} 1 & A(s) = o \\ 0 & A(s) \neq o \end{cases}$$

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Causal:  $P_A(o) = P(A \square \rightarrow o)$

$P$ (“if  $A$  were performed, outcome  $o$  would ensue”)

(Lewis, 1981)

# Dominance and Act-State Dependence

	$w_1$	$w_2$
<i>A</i>	1	3
<i>B</i>	2	4

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# Dominance and Act-State Dependence

Dominance reasoning is appropriate only when probability of outcome is *independent of choice*.

(A nasty nephew wants inheritance from his rich Aunt. The nephew wants the inheritance, but other things being equal, does not want to apologize. Does dominance give the nephew a reason to not apologize? *Whether or not the nephew is cut from the will may depend on whether or not he apologizes.*)



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**Law of Large Numbers:** everyone who maximizes expected utility will *almost certainly* be better off in the long run. By performing a random experiment sufficiently many times, the probability that the average outcome differs from the expected outcome can be rendered *arbitrarily* small.

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# Expected Utility Theory

R. A. Briggs. *Normative Theories of Rational Choice: Expected Utility*. Stanford Encyclopedia of Philosophy, 2014  
<https://plato.stanford.edu/entries/rationality-normative-utility/>.

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Representation theorems by Von Neumann-Morgenstern, Aumann-Anscombe, and Savage.



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- ▶ No action guidance. Rational decision makers do not prefer an act *because* its expected utility is favorable, but can only be described as *if* they were acting from this principle.
- ▶ Utility without chance. It seems odd from a linguistic point of view to say that the *meaning* of utility has something to do with preferences over lotteries.

# Issues

- ▶ Preference, choice and utility
- ▶ Preferences satisfy completeness and transitivity (Money-pump argument)
- ▶ Allais paradox: risk-aversion
- ▶ Ellsberg paradox: ambiguity-aversion
- ▶ Newcomb's paradox, Death in Damascus, Psychopath button problem, irrational choice: Act-state dependence
- ▶ Framing issues

# From Decisions to Games, I

Commenting on the difference between Robin Crusoe's maximization problem and the maximization problem faced by participants in a social economy, von Neumann and Morgenstern write:

“Every participant can determine the variables which describe his own actions but not those of the others. Nevertheless those “alien” variables cannot, from his point of view, be described by statistical assumptions.

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“Every participant can determine the variables which describe his own actions but not those of the others. Nevertheless those “alien” variables cannot, from his point of view, be described by statistical assumptions. This is because the others are guided, just as he himself, by rational principles—whatever that may mean—and no *modus procedendi* can be correct which does not attempt to understand those principles and the interactions of the conflicting interests of all participants.”

(vNM, pg. 11)

# Game Situations

1. a group of *self-interested* agents (players) involved in some interdependent decision problem

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	Bob	
<i>L</i>		<i>R</i>
	1	0

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# Game Situations

		Bob	
		<i>L</i>	<i>R</i>
Ann	<i>U</i>	1 1	0 0
	<i>D</i>	0 0	1 1

1. a **group** of **self-interested** agents (players) involved in some interdependent decision problem

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		Bob	
		<i>L</i>	<i>R</i>
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1. a *group* of *self-interested* agents (players) involved in some *interdependent* decision problem

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		<i>L</i>	<i>R</i>
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1. a group of *self-interested* agents (players) involved in some interdependent decision problem

# Just Enough Game Theory

A **game** is a mathematical model of a strategic interaction that includes

- ▶ the actions the players *can* take
- ▶ the players' interests (i.e., preferences),
- ▶ the “structure” of the decision problem

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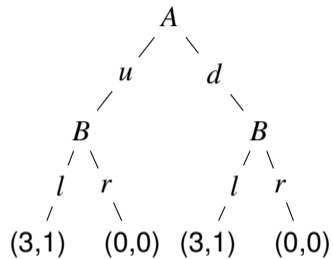
*It does not specify the actions that the players do take.*

# Games

		<i>B</i>	
		l	r
<i>A</i>	u	3, 1	0, 0
	d	0, 0	1, 3

# Games

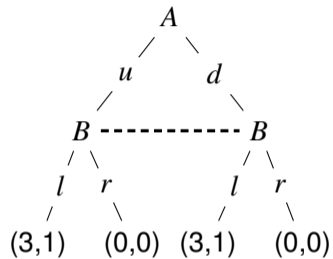
		<i>B</i>	
		<i>l</i>	<i>r</i>
<i>A</i>	<i>u</i>	3, 1	0, 0
	<i>d</i>	0, 0	1, 3





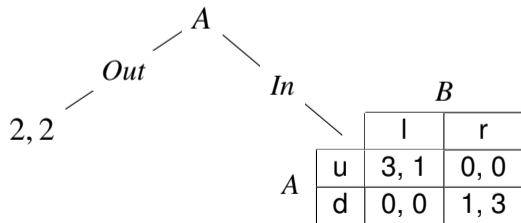
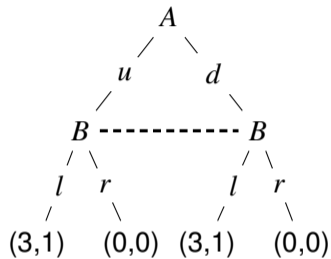
# Games

		<i>B</i>	
		<i>l</i>	<i>r</i>
<i>A</i>	<i>u</i>	3, 1	0, 0
	<i>d</i>	0, 0	1, 3



# Games

		<i>B</i>	
		<i>l</i>	<i>r</i>
<i>A</i>	<i>u</i>	3, 1	0, 0
	<i>d</i>	0, 0	1, 3



# From Decisions to Games, II

“*[T]he* fundamental insight of game theory [is] that a rational player must take into account that the players reason about each other in deciding how to play.”

R. Aumann and J. Dreze. *Rational Expectations in Games*. *American Economic Review*, 98, pp. 72-86, 2008.

# Solution Concept

A **solution concept** is a systematic description of the outcomes that may emerge in a family of games.

This is the starting point for most of game theory and includes many variants: Nash equilibrium, backwards induction, or iterated dominance of various kinds.

These are usually thought of as the embodiment of “rational behavior” in some way and used to analyze game situations.

Let  $G = \langle (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a finite strategic game (each  $S_i$  is finite and the set of players  $N$  is finite).

A **strategy profile** is an element  $\sigma \in S = S_1 \times \cdots \times S_n$

$\sigma$  is a (pure strategy) **Nash equilibrium** provided for all  $i$ , for all  $s_i \in S_i$ ,

$$u_i(\sigma) \geq u_i(s_i, \sigma_{-i})$$

# Zero-Sum Games

		Bob	
		<i>L</i>	<i>R</i>
Ann	<i>U</i>	1,4	4,1
	<i>D</i>	2,3	3,2

What should Ann *do*?

# Zero-Sum Games

		Bob	
		<i>L</i>	<i>R</i>
Ann	<i>U</i>	1,4	4,1
	<i>D</i>	2,3	3,2

What should Ann do? *Bob best choice in Ann's worst choice*

# Zero-Sum Games

		Bob		
		<i>L</i>	<i>R</i>	
Ann	<i>U</i>	1,4	4,1	1
	<i>D</i>	2,3	3,2	2

What should Ann do? *Security strategy: minimize over each row and choose the maximum value*



# Zero-Sum Games

		Bob	
		<i>L</i>	<i>R</i>
Ann	<i>U</i>	1,4	4,1
	<i>D</i>	2,3	3,2
		3	1

What should Bob do? *Security strategy: minimize over each column and choose the maximum value*

# Zero-Sum Games

		Bob	
		<i>L</i>	<i>R</i>
Ann	<i>U</i>	1,4	4,1
	<i>D</i>	2,3	3,2

The profile of security strategies  $(D, L)$  is a Nash equilibrium

# Matching Pennies

		Bob	
		<i>H</i>	<i>T</i>
Ann	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

There are no pure strategy Nash equilibria.

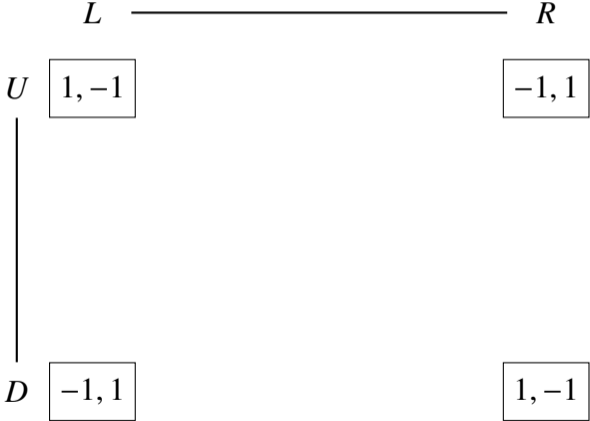
# Mixed Strategies

		Bob	
		<i>H</i>	<i>T</i>
Ann	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

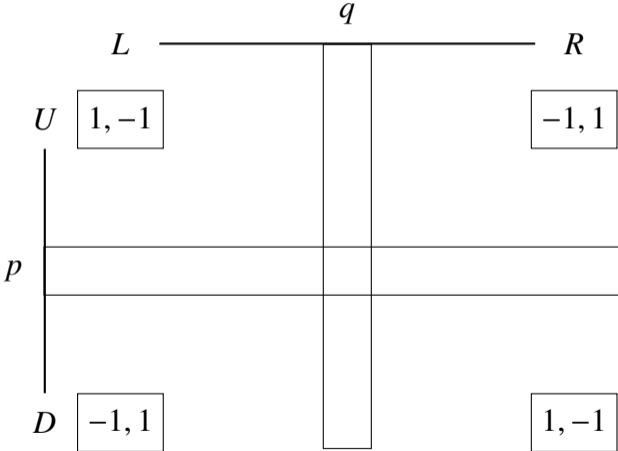
A **mixed strategy** is a probability distribution over the set of pure strategies. For instance:

- ▶  $[1/2 : H, 1/2 : T]$
- ▶  $[1/3 : H, 2/3 : T]$
- ▶ ...

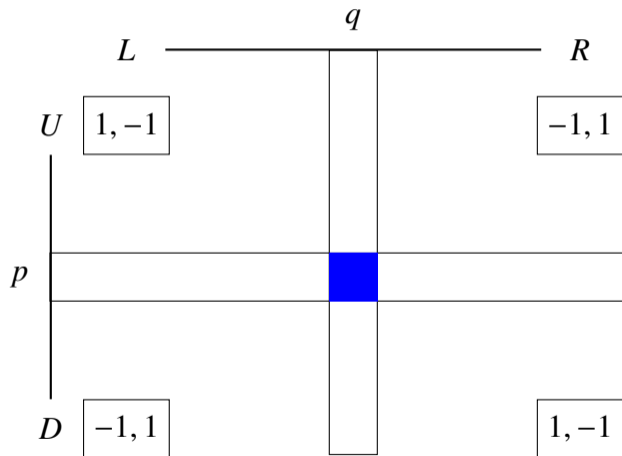
# Mixed Extension



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$$pq - p(1 - q) - (1 - p)q + (1 - p)(1 - q), -pq + p(1 - q) + (1 - p)q - (1 - p)(1 - q)$$

# Matching Pennies

		Bob	
		<i>H</i>	<i>T</i>
Ann	<i>H</i>	1, -1	-1, 1
	<i>T</i>	-1, 1	1, -1

The mixed strategy  $([1/2 : H, 1/2 : T], [1/2 : H, 1/2 : T])$  is the only Nash equilibrium.



**Theorem** (Von Neumann). For every two-player zero-sum game with finite strategy sets  $S_1$  and  $S_2$ , there is a number  $v$ , called the **value** of the game such that:

1.  $v = \max_{p \in \Delta(S_1)} \min_{q \in \Delta(S_2)} U_1(p, q) = \min_{q \in \Delta(S_2)} \max_{p \in \Delta(S_1)} U_1(p, q)$
2. The set of mixed Nash equilibria is nonempty. A mixed strategy profile  $(p, q)$  is a Nash equilibrium if and only if

$$p \in \operatorname{argmax}_{p \in \Delta(S_1)} \min_{q \in \Delta(S_2)} U_1(p, q)$$

$$q \in \operatorname{argmax}_{q \in \Delta(S_2)} \min_{p \in \Delta(S_1)} U_1(p, q)$$

3. For all mixed Nash equilibria  $(p, q)$ ,  $U_1(p, q) = v$

## In zero-sum games

- ▶ There exists a mixed strategy Nash equilibrium
- ▶ There may be more than one Nash equilibria
- ▶ Security strategies are always a Nash equilibrium
- ▶ Components of Nash equilibria are interchangeable: If  $\sigma$  and  $\sigma'$  are Nash equilibria in a 2-player game, then  $(\sigma_1, \sigma'_2)$  is also a Nash equilibrium.

Let  $G = \langle (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$  be a finite strategic game.

$$\Sigma_i = \{p \mid p : S_i \rightarrow [0, 1] \text{ and } \sum_{s_i \in S_i} p(s_i) = 1\}$$

The **mixed extension** of  $G$  is the game  $\langle (\Sigma_i)_{i \in N}, (U_i)_{i \in N} \rangle$  where for  $\sigma \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ :

$$U_i(\sigma) = \sum_{(s_1, \dots, s_n) \in S} \sigma_1(s_1) \sigma_2(s_2) \cdots \sigma_n(s_n) u_i(s_1, \dots, s_n)$$

**Theorem** (Nash). Every finite game  $G$  has a Nash equilibrium in mixed strategies (i.e., there is a Nash equilibrium in the mixed extension  $G$ ).

# Mixed Strategies

“We are reluctant to believe that our decisions are made at random. We prefer to be able to point to a reason for each action we take. Outside of Las Vegas we do not spin roulettes.” (Rubinstein)

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- ▶ One can think about a game as an interaction between large populations...a mixed strategy is viewed as the distribution of the pure choices in the population.

# Mixed Strategies

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- ▶ Mixed strategies are beliefs held by all *other* players concerning a player's actions.