Neighborhood Semantics for Modal Logic
Lecture 4

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Core Theory

- Neighborhood Semantics in the Broader Logical Landscape
-     Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

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Non-Normal Modal Logic with a Universal Modality

$(A\text{-}K)$	$A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi)$
$(A\text{-}T)$	$A\varphi \rightarrow \varphi$
$(A\text{-}4)$	$A\varphi \rightarrow AA\varphi$
$(A\text{-}B)$	$E\varphi \rightarrow AE\varphi$
$(A\text{-}Nec)$	From φ infer $A\varphi$
$(\langle\ ]\text{-}RM)$	From $\varphi \rightarrow \psi$ infer $\langle\ ]\varphi \rightarrow \langle\ ]\psi$
$(\langle\ ]\text{-}Cons)$	$\neg\langle\ ]\bot$
$(A\text{-}N)$	$A\varphi \rightarrow \langle\ ]\varphi$
$(Pullout)$	$\langle\ ](\varphi \wedge A\psi) \leftrightarrow (\langle\ ]\varphi \wedge A\psi)$

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Theorem. The logic EMA is sound and strongly complete with respect to neighborhood frames that are consistent, non-trivial and monotonic.

Filtrations

Let $\mathfrak{M} = \langle W, N, V \rangle$ be a neighborhood model and suppose that Σ is a set of sentences from $\mathcal{L}$.

For each $w, v \in W$, we say $w \sim_\Sigma v$ iff for each $\varphi \in \Sigma$, $w \models \varphi$ iff $v \models \varphi$.

For each $w \in W$, let $[w]_\Sigma = \{v \mid w \sim_\Sigma v\}$ be the equivalence class of $\sim_\Sigma$.

If $X \subseteq W$, let $[X]_\Sigma = \{[w] \mid w \in X\}$.

Filtrations

Definition

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and Σ a set of sentences closed under subformulas. A filtration of $\mathfrak{M}$ through Σ is a model $\mathfrak{M}^f = \langle W^f, N^f, V^f\rangle$ where

1. $W^f = [W]$
2. For each $w \in W$
   2.1 for each $\Box\varphi \in \Sigma$, $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$ iff $[[\![\varphi]\!]_{\mathfrak{M}}] \in N^f([w])$
3. For each $p \in \mathsf{At}$, $V(p) = [V(p)]$

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Theorem

Suppose that $\mathfrak{M}^f = \langle W^f, N^f, V^f \rangle$ is a filtration of $\mathfrak{M} = \langle W, N, V \rangle$ through (a subformula closed) set of sentences Σ. Then for each $\varphi \in \Sigma$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^f, [w] \models \varphi$

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3. For each $p \in \text{At}$, $V(p) = [V(p)]$

Corollary

E *has the finite model property. I.e., if* φ *has a model then there is a finite model.*

A Few Comments on Complexity

Logics without C (eg., $\mathbf{E}, \mathbf{EM}, \mathbf{E} + (\neg\Box\bot), \mathbf{E} + (\Box\varphi \to \Box\Box\varphi)$) are in NP.

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Logics with C are in PSPACE.

M. Vardi. *On the Complexity of Epistemic Reasoning*. IEEE (1989).

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J. Halpern and L. Rego. *Characterizing the NP-PSPACE gap in the satisfiability problem for modal logic*. Journal of Logic and Computation, 17:4, pgs. 795-806, 2007.

Can we import results/ideas from model theory for modal logic with respect to Kripke Semantics/Topological Semantics?

Frame Correspondence

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

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What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

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On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \rightarrow \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N\rangle$ *be a neighborhood frame such that for each* $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \to \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \to \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
3. $\Diamond\varphi \rightarrow \Box\varphi$
4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
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What about *augmented* frames?

Neighborhoods with nominals

$$p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid A\varphi$$

$p \in \text{At}$ and $i \in \text{Nom}$ (the set of nominals)

Neighborhood model with nominals $\langle W, N, V \rangle$,
$V : \text{At} \cup \text{Nom} \to \wp(W)$, where for all $i \in Nom$, $|V(i)| = 1$.

- $\mathfrak{M}, w \models i$ iff $V(w) = i$
- $\mathfrak{M}, w \models A\varphi$ iff for all $v \in W$, $\mathfrak{M}, v \models \varphi$

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- $\mathfrak{M}, w \models A\varphi$ iff for all $v \in W$, $\mathfrak{M}, v \models \varphi$

$$\text{(BG)} \qquad \frac{\vdash (i \wedge \Diamond j) \rightarrow E(j \wedge \varphi)}{\vdash E(i \wedge \Box\varphi)}$$

for $i \neq j$ and j not occurring in φ

Characterizing Augmented Frames

Theorem. A neighborhood frame is augmented iff it *admits** the rule BG.

B. ten Cate and T. Litak. *Topological Perspective on Hybrid Proof Rules*. Electronic Notes in Theoretical Computer Science, 174, pgs. 79 - 94, 2007.

* A class of frames admits a rule provided every falsifying model of the consequent can be *extended* to a falsifying model of the premises.

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

Definition

Given a neighborhood model $\mathcal{M} = \langle W, N, V\rangle$, define a Kripke model $\mathcal{M}^\circ = \langle V, R_N, R_{\not\ni}, R_N, Pt, V\rangle$ as follows:

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Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathrm{Pt}$$

where $p \in \mathrm{At}$ and Pt is a unary modal operator.

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Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \textit{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$

Monotonic Models

Lemma

On Monotonic Models $\langle N \rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N \rangle([\ni]ST(\varphi))$

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in Proof Theory of Modal Logic, Kluwer, pgs. 293 - 311, 1996.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. The Journal of Symbolic Logic, 64:1, pgs. 99 - 138, 1999.

Model/Frame Constructions

Disjoint Union

Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models. The **disjoint union of $\mathfrak{M}_1$ and $\mathfrak{M}_2$** is the neighborhood model $\mathfrak{M}_1 + \mathfrak{M}_2 = \langle W_1 + W_2, N, V \rangle$ where for all $p \in \text{At}$, $V(p) = V_1(p) \cup V_2(p)$; and for $i = 1, 2$,

for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

(Similar definition for frames)

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for all $X \subseteq W_1 + W_2$, and $w \in W_i$, $X \in N(w)$ iff $X \cap W_i \in N_i(w)$.

(Similar definition for frames)

Proposition. For all $\varphi \in \mathcal{L}$, for $i = 1, 2$, if $w \in W_i$, then $\mathfrak{M}_1 + \mathfrak{M}_2, w \models \varphi$ iff $\mathfrak{M}_i, w \models \varphi$.

Fact. The universal modality is not definable in the basic modal language.

Monotonic Bisimulation

Let $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ be two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a bisimulation provided whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: If $X \in N(w)$ then there is an $X' \subseteq W'$ such that

$$X' \in N'(w') \text{ and } \forall x' \in X' \; \exists x \in X \text{ such that } xZx'$$

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Lemma

On locally core-finite models, if $\mathfrak{M}, w \equiv_{\mathcal{L}} \mathfrak{M}', w'$ then $\mathfrak{M}, w \;\underline{\leftrightarrow}\; \mathfrak{M}', w'$.

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Theorem

A first-order formula (in the appropriate language...) $\alpha(x)$ is invariant for monotonic bisimulation, then $\alpha(x)$ is equivalent to $st_x^{mon}(\varphi)$ for some $\varphi \in \mathcal{L}$.

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M. Pauly. *Bisimulation for Non-normal Modal Logic*. 1999.

H. Hansen. *Monotonic Modal Logic*. 2003.

Do monotonic bisimulations work when we drop monotonicity? No!

Bounded Morphisms

If $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

for all $X \subseteq W_2$, we have $f^{-1}[X] \in N_1(w)$ iff $X \in N_2(f(w))$;

and for all $p \in \mathsf{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

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Lemma Let $\mathfrak{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathfrak{M}_1, w \models \varphi$ iff $\mathfrak{M}_2, f(w) \models \varphi$.

Definition

Two points w_1 from $\mathfrak{F}_1$ and w_2 from $\mathfrak{F}_2$ are behavorially equivalent provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \rightarrow \mathfrak{F}$ and $g : \mathfrak{F}_2 \rightarrow \mathfrak{F}$ such that $f(w_1) = g(w_2)$.

Theorem

Over the class **N** *(of neighborhood models), the following are equivalent:*

- $\alpha(x)$ *is equivalent to the translation of a modal formula*
- $\alpha(x)$ *is invariant under behavioural equivalence.*

H. Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5(2:2), pgs. 1 - 38, 2009.

The Language $\mathcal{L}_2$

The language $\mathcal{L}_2$ is built from the following grammar:

$$x = y \mid u = v \mid \mathsf{P}_i x \mid x\mathsf{N}u \mid u\mathsf{E}x \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists u\varphi$$

$\mathfrak{M} = \langle D, \{P_i \mid i \in \omega\}, N, E\rangle$ where

- $D = D^{\mathsf{s}} \cup D^{\mathsf{n}}$ (and $D^{\mathsf{s}} \cap D^{\mathsf{n}} = \emptyset$),
- $Q_i \subseteq D^{\mathsf{s}}$,
- $N \subseteq D^{\mathsf{s}} \times D^{\mathsf{n}}$ and
- $E \subseteq D^{\mathsf{n}} \times D^{\mathsf{s}}$.

The Language $\mathcal{L}_2$

Definition

Let $\mathfrak{M} = \langle S, N, V \rangle$ be a neighbourhood model. The *first-order translation* of $\mathcal{M}$ is the structure $\mathfrak{M}^\circ = \langle D, \{P_i \mid i \in \omega\}, R_N, R_\ni \rangle$ where

- $D^{\mathsf{s}} = S$, $D^{\mathsf{n}} = \bigcup_{s \in S} N(s)$
- For each $i \in \omega$, $P_i = V(p_i)$
- $R_N = \{(s, U) \mid s \in D^{\mathsf{s}}, U \in N(s)\}$
- $R_\ni = \{(U, s) \mid s \in D^{\mathsf{s}}, s \in U\}$

The Language $\mathcal{L}_2$

Definition

The *standard translation* of the basic modal language are functions $st_x : \mathcal{L} \rightarrow \mathcal{L}_2$ defined as follows as follows: $st_x(p_i) = \mathsf{P}_i x$, st_x commutes with boolean connectives and

$$st_x(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_\ni y \leftrightarrow st_y(\varphi)))$$

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$$st_x(\Box\varphi) = \exists u(x\mathsf{R}_N u \wedge (\forall y(u\mathsf{R}_\ni y \leftrightarrow st_y(\varphi)))$$

Lemma

Let $\mathfrak{M}$ be a neighbourhood structure and $\varphi \in \mathcal{L}$. For each $s \in S$, $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}^\circ \models st_x(\varphi)[s]$.

$\mathbf{N} = \{\mathfrak{M} \mid \mathfrak{M} \cong \mathfrak{M}^\circ \text{ for some neighbourhood model } \mathfrak{M}\}$

$$
\begin{aligned}
&\text{(A1)} \quad \exists x(x = x) \\
&\text{(A2)} \quad \forall u \exists x(x \mathsf{R}_N u) \\
&\text{(A3)} \quad \forall u, v(\neg(u = v) \rightarrow \\
&\qquad\qquad \exists x((u \mathsf{R}_\ni x \wedge \neg v \mathsf{R}_\ni x) \vee (\neg u \mathsf{R}_\ni x \wedge v \mathsf{R}_\ni x)))
\end{aligned}
$$

Theorem

Suppose $\mathfrak{M}$ is an $\mathcal{L}_2$-structure. Then there is a neighbourhood structure $\mathfrak{M}_\circ$ such that $\mathfrak{M} \cong (\mathfrak{M}_\circ)^\circ$.

Theorem

Over the class **N** *(of neighborhood models), the following are equivalent:*

- $\alpha(x)$ *is equivalent to the translation of a modal formula*
- $\alpha(x)$ *is invariant under behavioural equivalence.*

H. Hansen, C. Kupke and EP. *Neighbourhood Structures: Bisimilarity and Basic Model Theory*. Logical Methods in Computer Science, 5(2:2), pgs. 1 - 38, 2009.

Course Plan

✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics

✓ **Core Theory**: Relationship with Other Semantics for Modal Logic, Model Theory; Completeness, Decidability, Complexity, Incompleteness

1. **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods

Neighborhood Models for First-Order Modal Logic

H. Arlo Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

Higher-Order Coalition Logic (time permitting)

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 2010.

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

State-of-the-art

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D.Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Draft available (2008).

`http://lpcs.math.msu.su/~shehtman/QNCLfinal.pdf`

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x-**variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

First-order Modal Logic

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Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in V(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq (\varphi)^{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D \rangle$ where W and D are sets, and $N : W \to \wp\wp W$.

A **constant domain Neighborhood model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $V(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in V(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $(\varphi)^{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

Let **S** be any (classical) propositional modal logic, by **FOL** + **S** we mean the set of formulas closed under the following rules and axioms:

(S) All instances of axioms and rules from **S**.

(∀) $\forall x A \rightarrow A^x_t$ (where t is free for x in A)

(Gen) $\dfrac{A \rightarrow B}{A \rightarrow \forall x B}$, where x is not free in A.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Barcan Schemas

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- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

Observation 1: CBF is provable in $\mathbf{FOL} + \mathbf{EM}$

Observation 2: BF and CBF both valid on relational frames with constant domains

Observation 3: BF is valid in a *varying* domain relational frame iff the frame is anti-monotonic; CBF is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the '**lottery paradox**':

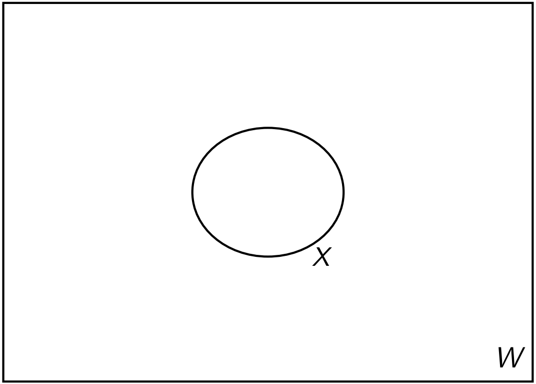
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

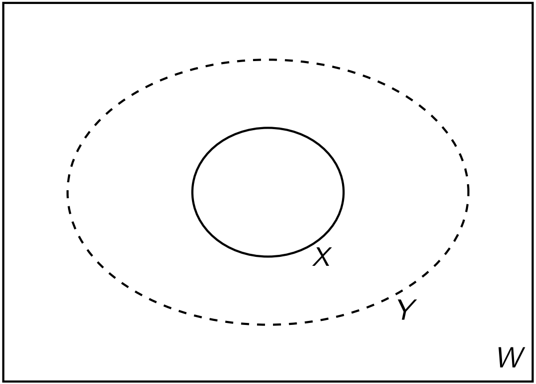
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

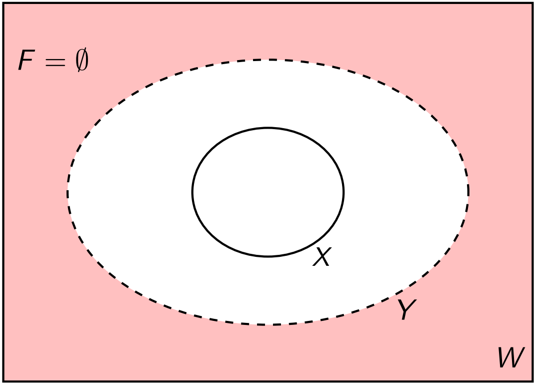
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



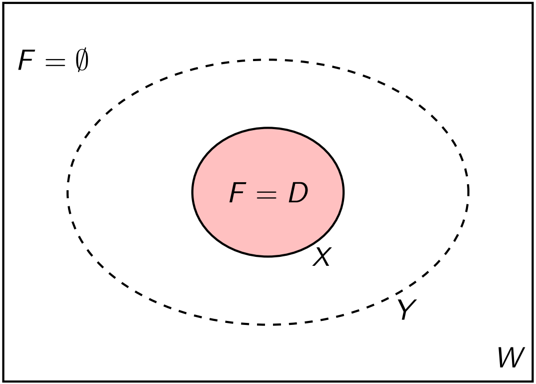
$$X \in N(w)$$



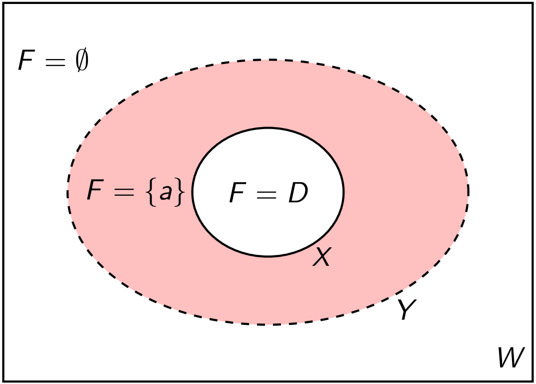
$Y \notin N(w)$



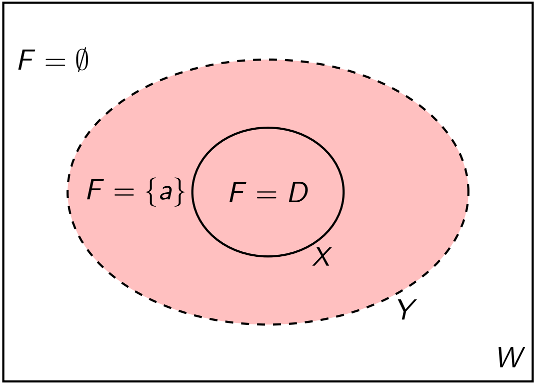
$$\forall v \notin Y, \quad I(F, v) = \emptyset$$



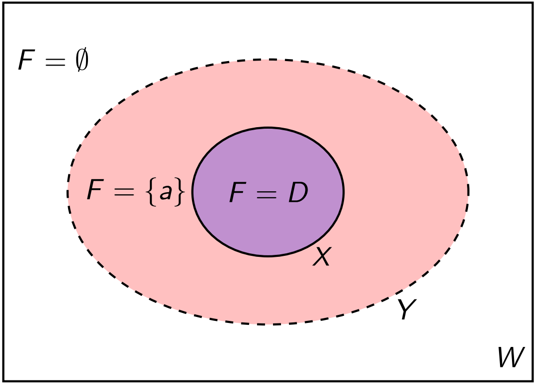
$$\forall v \in X, \quad I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \quad I(F, v) = D = \{a\}$$



$$(F[a])^{\mathcal{M}} = Y \notin N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x)$$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

hence $\quad w \models \Box \forall x F(x)$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Completeness Theorems

Theorem FOL $+$ **E** is sound and strongly complete with respect to the class of **all** frames.

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Theorem FOL $+$ **E** is sound and strongly complete with respect to the class of **all** frames.

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Theorem FOL $+$ **EM** is sound and strongly complete with respect to the class of supplemented frames.

Theorem FOL $+$ **E** $+$ CBF is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

FOL + **K** and **FOL** + **K** + *BF*

Theorem **FOL** + **K** is sound and strongly complete with respect to the class of filters.

FOL + **K** and **FOL** + **K** + *BF*

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Observation The augmentation of the smallest canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** + **K** is not a canonical model for **FOL** + **K**.

FOL $+$ **K** and **FOL** $+$ **K** $+$ *BF*

Theorem **FOL** $+$ **K** is sound and strongly complete with respect to the class of filters.

Observation The augmentation of the smallest canonical model for **FOL** $+$ **K** is not a canonical model for **FOL** $+$ **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** $+$ **K** is not a canonical model for **FOL** $+$ **K**.

Lemma The augmentation of the smallest canonical model for **FOL** $+$ **K** $+$ *BF* is a canonical for **FOL** $+$ **K** $+$ *BF*.

Theorem **FOL** $+$ **K** $+$ *BF* is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

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1. **S4M** is complete for the class of all frames that are reflexive, transitive and *final* (every world can see an 'end-point'). However **FOL** + **S4M** is incomplete for Kripke models based on **S4M**-frames. (see Hughes and Cresswell, pg. 283).

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2. **S4.2** is S4 with $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$. This logics is complete for the class of frames that are reflexive, transitive and *convergent*. However, **FOL** + **S4M** + *BF* is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Creswell, pg. 271)

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3. The quantified extension of **GL** is not complete (with respect to varying domains models).

What is going on?

R. Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics*. Lecture Notes in Logic No. 38, Cambridge University Press, 2011.

An Application: Coalition Logic

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 19th European Conference on Artificial Intelligence, pgs. 555 - 560, 2010.

▸ Skip

Q. Chen and K. Su. *Higher-Order Epistemic Coalition Logic for Multi-Agent Systems*. 7th Workshop on Logical Aspects of Multi-Agent Systems, 2014.

Coalition Logic: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [C]\varphi$

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$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^{\mathcal{M}} \in N(w, C)$: "Coalition C has a joint strategy to force the outcome to satisfy φ".

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Higher-Order Coalition Logic: $\varphi :=$
$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

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$F(x_1, \ldots, x_n) \mid Xx \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall X\varphi \mid \forall x\varphi \mid [\{x\}\varphi]\varphi \mid \langle\{x\}\varphi\rangle\varphi$

- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- x is a first-order variable
- X is a set variable
- $\{x\}\psi$ is a group operator representing the set of all d such that $\psi[d/x]$ holds

HCL: Expressivity

What does the added expressive power give you?

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  $\forall X(\forall x(Xx \rightarrow user(x)) \rightarrow [\{y\}Xy]\varphi)$

  Every coalition such that all of its members are users can achieve φ.

HCL: Expressivity

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- Relationships between coalitions:
  $\forall x(\mathit{super\_user}(x) \rightarrow \mathit{user}(x))$

- General quantification over coalitions:
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  Every coalition such that all of its members are users can achieve φ*.*

- Complex relationships between coalitions and agents:
  $[\{x\}\varphi(x)]\psi \rightarrow [\{y\}\exists x(\varphi(x) \wedge \mathit{collaborates}(y,x))]\psi$

  If the coalition represented by φ *can achieve* ψ *then so can any group that collaborates with at least one member of* $\varphi(x)$*.*

HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)$

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Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \to [\{x\}\varphi(x)]\forall y\psi(y)$

$[\{x\}x = Eric]\forall y(ESSLLI(y) \to happy(y)) \to$
$\forall y[\{x\}x = Eric](ESSLLI(y) \to happy(y))$

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If I can do something to make everyone happy at ESSLLI implies for each person at ESSLLI, I can do something to make them happy.

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$\forall y[\{x\}x = \mathit{Eric}](\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y)) \not\rightarrow$
$[\{x\}x = \mathit{Eric}]\forall y(\mathit{ESSLLI}(y) \rightarrow \mathit{happy}(y))$

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$\forall y[\{x\}x = Eric](ESSLLI(y) \rightarrow happy(y)) \not\rightarrow$
$[\{x\}x = Eric]\forall y(ESSLLI(y) \rightarrow happy(y))$

For each person at ESSLLI, I can make them happy does not imply that I can do something to make everyone at ESSLLI happy.

Higher-Order Coalition Logic

Sound and complete axiomatization combines ideas from coaltion logic, first-order extensions of non-normal modal logics and Henkin-style completeness for second-order logic.

Neighborhood semantics in action

Background: Propositional Dynamic Logic

Let P be a set of atomic programs and At a set of atomic propositions.

Formulas of **PDL** have the following syntactic form:

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid [\alpha]\varphi$$

$$\alpha := a \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \varphi?$$

where $p \in$ At and $a \in$ P.

$[\alpha]\varphi$ is intended to mean "after executing the program α, φ is true"

Background: Propositional Dynamic Logic

Semantics: $\mathcal{M} = \langle W, \{R_a \mid a \in \mathrm{P}\}, V \rangle$ where for each $a \in \mathrm{P}$, $R_a \subseteq W \times W$ and $V : \mathrm{At} \rightarrow \wp(W)$

- $R_{\alpha \cup \beta} := R_\alpha \cup R_\beta$
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$
- $R_{\alpha^*} := \cup_{n \geq 0} R_\alpha^n$
- $R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$

$\mathcal{M}, w \models [\alpha]\varphi$ iff for each v, if $wR_\alpha v$ then $\mathcal{M}, v \models \varphi$

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
7. $\varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi$
8. Modus Ponens and Necessitation (for each program α)

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$ (Fixed-Point Axiom)
7. $\varphi \wedge [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow [\alpha^*]\varphi$ (Induction Axiom)
8. Modus Ponens and Necessitation (for each program α)

Background: Propositional Dynamic Logic

Theorem PDL is sound and weakly complete with respect to the Segerberg Axioms.

Theorem The satisfiability problem for **PDL** is decidable ($\mathrm{EXPTIME}$-Complete).

D. Kozen and R. Parikh. *A Completeness proof for Propositional Dynamic Logic*.
.

D. Harel, D. Kozen and Tiuryn. *Dynamic Logic*. 2001.

Concurrent Programs

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

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Concurrent Programs

$\alpha \cap \beta$ is intended to mean "execute α and β in parallel".

In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

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Concurrent Programs

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In PDL: $R_\alpha \subseteq W \times W$, where $wR_\alpha v$ means executing α in state w leads to state v.

With Concurrent Programs: $R_\alpha \subseteq W \times \wp(W)$, where $wR_\alpha V$ means executing α in parallel from state w to reach all states in V.

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With Concurrent Programs: $R_\alpha \subseteq W \times \wp(W)$, where $wR_\alpha V$ means executing α in parallel from state w to reach all states in V.

$w \models \langle\alpha\rangle\varphi$ iff $\exists U$ such that $(w, U) \in R_\alpha$ and $\forall v \in U$, $v \models \varphi$.

$R_{\alpha\cap\beta} := \{(w, V) \mid \exists U, U',\ (w, U) \in R_\alpha,\ (w, U') \in R_\beta,\ V = U \cup U'\}$

D. Peleg. *Concurrent Dynamic Logic*. JACM (1987).

From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

From **PDL** to Game Logic

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Main Idea:
In **PDL**: $w \models \langle \pi \rangle \varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

Main Idea:
In **PDL**: $w \models \langle \pi \rangle \varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

Game Logic generalized **PDL** by considering two players:

In **GL**: $w \models \langle \gamma \rangle \varphi$: Angel has a **strategy** in the game γ to ensure that the game ends in a state where φ is true.

From **PDL** to Game Logic

Consequences of two players:

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However, $[\gamma]\varphi \wedge [\gamma]\psi \rightarrow [\gamma](\varphi \wedge \psi)$ is **not** a valid principle

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
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Game Logic

Syntax

Let Γ_0 be a set of atomic games and At a set of atomic propositions. Then formulas of Game Logic are defined inductively as follows:

$$\gamma \ := \ g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$

$$\varphi \ := \ \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \mid [\gamma]\varphi$$

where $p \in \mathsf{At}, g \in \Gamma_0$.

Game Logic

A neighborhood game model is a tuple
$\mathcal{M} = \langle W, \{E_g \mid g \in \Gamma_0\}, V\rangle$ where

W is a nonempty set of states

For each $g \in \Gamma_0$, $E_g : W \rightarrow \wp(\wp(W))$ is a monotonic neighborhood function.

$X \in E_g(w)$ means in state s, Angel has a strategy to force the game to end in *some* state in X (we may write wE_gX)

$V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function.

Game Logic

Propositional letters and boolean connectives are as usual.

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $(\varphi)^{\mathcal{M}} \in E_\gamma(w)$

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$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $(\varphi)^{\mathcal{M}} \in E_\gamma(w)$

Suppose $E_\gamma(Y) := \{s \mid Y \in E_g(s)\}$

- $E_{\gamma_1;\gamma_2}(Y) \;:=\; E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1 \cup \gamma_2}(Y) \;:=\; E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\varphi?}(Y) \;:=\; (\varphi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) \;:=\; \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) \;:=\; \mu X.Y \cup E_\gamma(X)$

Game Logic: Axioms

1. All propositional tautologies
2. $\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$ Composition
3. $\langle\alpha\cup\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi$ Union
4. $\langle\psi?\rangle\varphi \leftrightarrow (\psi\wedge\varphi)$ Test
5. $\langle\alpha^{d}\rangle\varphi \leftrightarrow \neg\langle\alpha\rangle\neg\varphi$ Dual
6. $(\varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi) \rightarrow \langle\alpha^*\rangle\varphi$ Mix

and the rules,

$$\frac{\varphi \qquad \varphi\rightarrow\psi}{\psi} \qquad \frac{\varphi\rightarrow\psi}{\langle\alpha\rangle\varphi\rightarrow\langle\alpha\rangle\psi} \qquad \frac{(\varphi\vee\langle\alpha\rangle\psi)\rightarrow\psi}{\langle\alpha^*\rangle\varphi\rightarrow\psi}$$

Game Logic

- Game Logic is more expressive than **PDL**

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$$\langle (g^d)^* \rangle \bot$$

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

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Open Question Is (full) game logic complete with respect to the class of all game models?

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M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001)..

Game Logic

Theorem Given a game logic formula φ and a finite game model $\mathcal{M}$, model checking can be done in time $O(|\mathcal{M}|^{ad(\varphi)+1} \times |\varphi|)$

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Game Logic

Theorem The satisfiability problem for game logic is in EXPTIME.

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Game Logic

Theorem Game logic can be translated into the modal μ-calculus

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Theorem Game logic can be translated into the modal μ-calculus

Theorem No finite level of the modal μ-calculus hierarchy captures the expressive power of game logic.

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Game Algebra

Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models

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Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models (iff $\langle\gamma_1\rangle p \leftrightarrow \langle\gamma_2\rangle p$ is valid for a p which occurs neither in γ_1 nor in γ_2.)

Game Algebra

Game Boards: Given a set of states or positions B, for each game g and each player i there is an associated relation $E_g^i \subseteq B \times 2^B$:

$pE_g^i T$ holds if in position p, i can force that the outcome of g will be a position in T.

- (monotonicity) if $pE_g^i T$ and $T \subseteq U$ then $pE_g^i U$
- (consistency) if $pE_g^i T$ then not $pE_g^{1-i}(B - T)$

Given a game board (a set B with relations E_g^i for each game and player), we say that two games g, h ($g \approx h$) are equivalent if $E_g^i = E_h^i$ for each i.

Game Algebra

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5. $y \preceq z \Rightarrow x; y \preceq x; z$

Theorem Sound and complete axiomatizations of (iteration free) game algebra

Y. Venema. *Representing Game Algebras*. Studia Logica **75** (2003)..

V. Goranko. *The Basic Algebra of Game Equivalences*. Studia Logica **75** (2003)..

Concurrent Game Logic

$\gamma_1 \cap \gamma_2$ means "play γ_1 and γ_2 in parallel."

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Need both the disjunctive and conjunctive interpretation of the neighborhoods.

Main Idea: $R_\gamma \subseteq W \times \wp(\wp(\wp(W)))$

J. van Benthem, S. Ghosh and F. Liu. *Modelling Simultaneous Games in Dynamic Logic*. Synthese, 165(2), pgs. 247-268, 2008.

More Information on Game Logic and Algebra

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica **75**, 2003.

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics, 1985.

J. van Benthem. *Logics and Games*. The MIT Press, 2014.

Background: Modeling Informational Changes

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Accept evidence from an infallible source.

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Let $\mathcal{M} = \langle W, E, V\rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi}\rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathsf{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

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$[!\varphi]\psi$: "ψ is true after the public announcement of φ"

$\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}^{!\varphi}, w \models \psi$

Public Announcements: Recursion Axioms

$$
\begin{array}{lcl}
[!\varphi]p & \leftrightarrow & (\varphi \rightarrow p) \qquad (p \in \mathsf{At}) \\
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[!\varphi]\neg\psi & \leftrightarrow & (\varphi \rightarrow \neg[!\varphi]\psi) \\
[!\varphi]\Box\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi}[!\varphi]\psi) \\
[!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\
[!\varphi]\Box^{\alpha}\psi & \leftrightarrow & (\varphi \rightarrow \Box^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi) \\
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\end{array}
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1. **Evidence addition**: accepting that φ is a piece of evidence
2. **Evidence removal**: remove evidence for $\neg\varphi$
3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Addition

Let $\mathcal{M} = \langle W, E, V\rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi}\rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

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Evidence Addition: Recursion Axioms

$$[+\varphi]\Box\psi \leftrightarrow (E\varphi \rightarrow (\Box[+\varphi]\psi \vee A(\varphi \rightarrow [+\varphi]\psi)))$$

$$[+\varphi]\Box^{\alpha}\psi \leftrightarrow (E\varphi \rightarrow (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge A((\varphi \wedge [+\varphi]\alpha) \rightarrow [+\varphi]\psi))))$$

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$$[+\varphi]\Box^{\alpha}\psi \quad \leftrightarrow \quad (E\varphi \rightarrow (\Box^{[+\varphi]\alpha}[+\varphi]\psi \vee (E(\varphi \wedge [+\varphi]\alpha) \wedge A((\varphi \wedge [+\varphi]\alpha) \rightarrow [+\varphi]\psi))))$$

Evidence Addition: Recursion Axioms

$$[+\varphi]B\psi \quad \leftrightarrow \quad ????$$

$$[+\varphi]B^{\alpha}\psi \quad \leftrightarrow \quad ????$$

Adding φ

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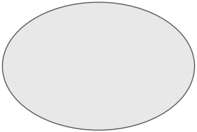
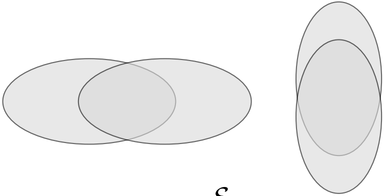
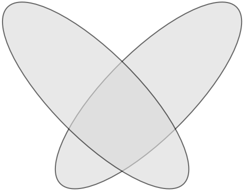


$\mathcal{E}_1$

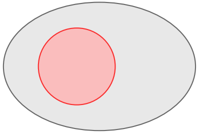


$\mathcal{E}_2$

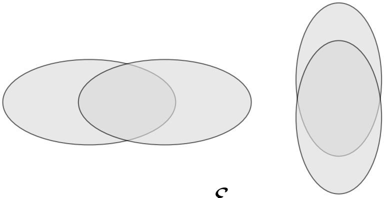


$\mathcal{E}_3$

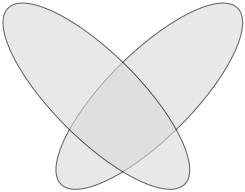
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2$



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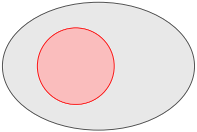
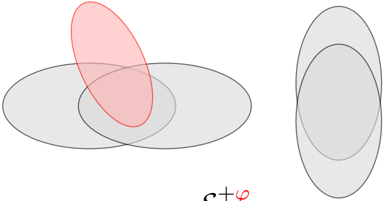
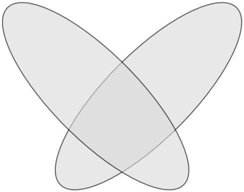


$\mathcal{E}_1^{+\varphi}$

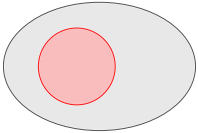


$\mathcal{E}_2^{+\varphi}$

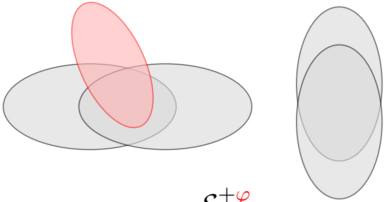


$\mathcal{E}_3$

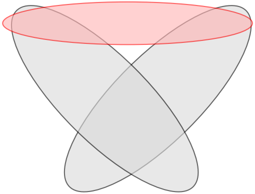
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2^{+\varphi}$



$\mathcal{E}_3^{+\varphi}$

Compatibile vs. Incompatible

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally **φ-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

Compatibile vs. Incompatible

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2. $\mathcal{X}$ is **incompatible** with φ provided there are $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \cdots \cap X_n \subseteq [\![\neg\varphi]\!]_{\mathcal{M}}$.

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Conditional belief: $B^{+\varphi}\psi$ iff for each maximally φ-compatible $\mathcal{X} \subseteq E(w)$, $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$

Conditional Beliefs (Incompatibility Version): $\mathcal{M}, w \models B^{-\varphi}\psi$ iff for all maximal f.i.p., if $\mathcal{X}$ is incompatible with φ then $\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

$B^{+\neg\varphi}$ vs. $B^{-\varphi}$

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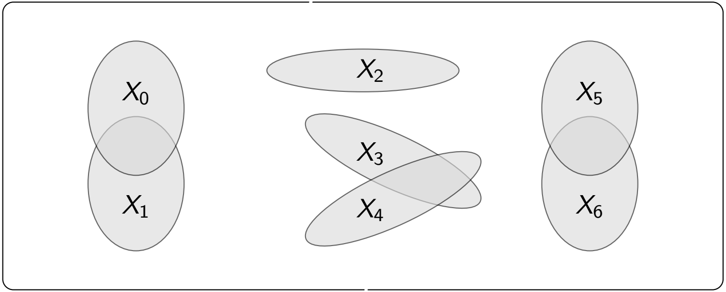

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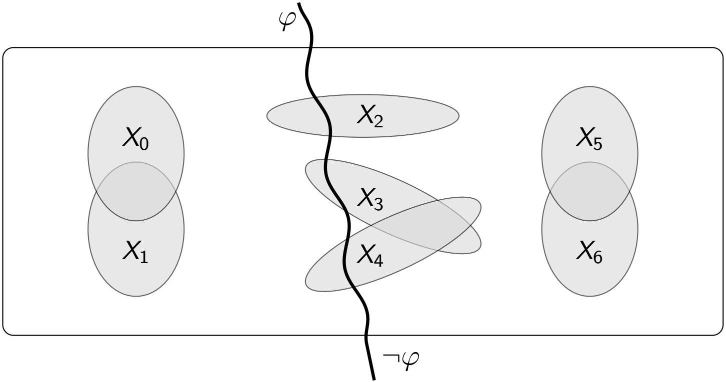

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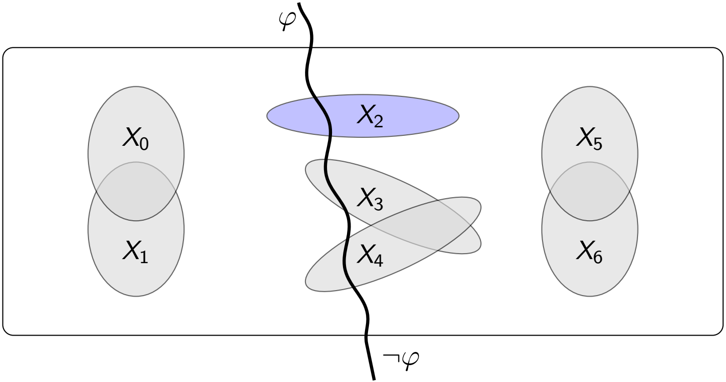

$\{X_2\}$ is (max.) compatible with $\neg\varphi$ but not maximally φ incompatible

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

▸ Proof Sketch

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Language Extension: $\mathcal{M}, w \models B^{\varphi,\psi}\chi$ iff for all maximally φ-compatible sets $\mathcal{X} \subseteq E(w)$, if $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$, then $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\chi]\!]_{\mathcal{M}}$.

$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

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$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

Fact. The following is valid:

$$[+\varphi]B^{\psi,\alpha}\chi \leftrightarrow (E\varphi \rightarrow (B^{\varphi\wedge[+\varphi]\psi,[+\varphi]\alpha}[+\varphi]\chi \wedge B^{[+\varphi]\psi,\neg\varphi\wedge[+\varphi]\alpha}[+\varphi]\chi))$$

Dissecting the Public Announcement Operation

On evidence models, a **public announcement** ($!\varphi$) is a complex combination of three distinct epistemic operations:

✓ **Evidence addition**: accepting that φ is a piece of evidence

2. **Evidence removal**: remove evidence for $\neg\varphi$

3. **Evidence modification**: incorporate φ into each piece of evidence gathered so far

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$ ▸ More

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Evidence Modification: $E^{\oplus\varphi}(w) = \{X \cup [\![\varphi]\!]_{\mathcal{M}} \mid X \in E(w)\}$

$\mathcal{M}, w \models [\oplus\varphi]\psi$ iff $\mathcal{M}^{\oplus\varphi}, w \models \psi$

- $[\oplus\varphi]\Box\psi \leftrightarrow (\Box[\oplus\varphi]\psi \wedge A(\varphi \rightarrow [\oplus\varphi]\psi))$

Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

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$\mathcal{M}, w \models [\oplus\varphi]\psi$ iff $\mathcal{M}^{\oplus\varphi}, w \models \psi$

- $[\oplus\varphi]\Box\psi \leftrightarrow (\Box[\oplus\varphi]\psi \wedge A(\varphi \rightarrow [\oplus\varphi]\psi))$

Evidence Combination: $E^{\#}(w)$ is the smallest set closed under consistent intersection and containing $E(w)$

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$

- Are $\neg[\#]\Box\neg\varphi \rightarrow B\varphi$ and $[\#]\Box\varphi \rightarrow B\varphi$ valid? ▸ Explain

Summary: Conditional Belief/Evidence

$\Box\psi$:	"there is evidence for ψ"
$\Box^{\varphi}\psi$:	"there is evidence compatible with φ for ψ"
$\Box_{\overline{\gamma}}\psi$:	"there is evidence compatible with each of the γ_i for ψ"

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Complete logical analysis?

$$B^{\varphi}\psi \to B(\varphi \to \psi) \quad \text{and} \quad B(\varphi \to \psi) \to B^{\top,\varphi}\psi$$

Summary: Evidence Operations

Public announcement:	$[!\varphi]B\psi$	$\leftrightarrow$	$(\varphi \rightarrow B^{\varphi}[!\varphi]\psi)$
Evidence addition:	$[+\varphi]B\psi$	$\leftrightarrow$	$(E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$
Evidence removal:	$[-\varphi]B\psi$	$\leftrightarrow$	$(\neg A\varphi \rightarrow B_{\neg\varphi}[-\varphi]\psi)$

Concluding Remarks

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Robust Belief: $\mathcal{M}, w \models B^r\varphi$ iff for each $X \subseteq W$ with $w \in X$, we have $Min_{\preceq}(X) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

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Reliable Evidence: $E^C(w) = \{X \in E(w) \mid w \in X\}$

$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

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$\mathcal{M}, w \models \Box^C\varphi$ iff for all $v \in \bigcap E^C(w)$, $\mathcal{M}, v \models \varphi$

Unreliable Evidence: $E^U(w) = \{X \in E(w) \mid w \notin X\}$.

$\mathcal{M}, w \models \Box^U\varphi$ iff for all $v \in \bigcup E^U(w)$, $\mathcal{M}, v \models \varphi$

Concluding Remarks: Robust Belief, Reliable and Unreliable Evidence

Fact. Let $\mathcal{M}$ be a uniform evidence model, then for all factual formulas φ:

$$\mathcal{M}, w \models \Box^{C}\varphi \wedge \Box^{U}\varphi \text{ iff } ORD(\mathcal{M}), w \models B^{r}\varphi$$

▸ Explain

Fact The operators $\Box^{C}$ and $\Box^{U}$ are not definable in evidence belief language $\mathcal{L}$. ▸ Proof

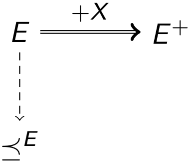
Concluding Remarks: Evidence Addition and Plausibility

$$E$$

$$\Big\downarrow$$

$$\preceq^E$$

Concluding Remarks: Evidence Addition and Plausibility

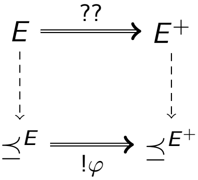


Concluding Remarks: Evidence Addition and Plausibility

$$\begin{array}{ccc} E & \xrightarrow{+X} & E^+ \\ \downarrow & & \downarrow \\ \preceq^E & \xrightarrow[??]{} & \preceq^{E^+} \end{array}$$

$$\preceq^{E^+} = \preceq^E - \{(w, v) \mid v \in X \text{ and } w \notin X\}.$$

Concluding Remarks: Evidence Addition and Plausibility



Concluding Remarks: Many Agents

Social notions: Let $\mathcal{M} = \langle W, \mathcal{E}_i, \mathcal{E}_j, V\rangle$ be a multiagent evidence model. What evidence does the group i, j have?

- $\mathcal{M}, w \models \Box^{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cup \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models \Box_{\{i,j\}}\varphi$ iff there is a $X \in \mathcal{E}_i \cap \mathcal{E}_j$ such that $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [i \sqcap j]\varphi$ iff there exists $X \in \mathcal{E}_i \sqcap \mathcal{E}_j$ with $X \subseteq [\![\varphi]\!]_{\mathcal{M}}$
  $\mathcal{E}_i \sqcap \mathcal{E}_j = \{Y \mid \emptyset \neq Y = X \cap X'$ with $X \in \mathcal{E}_i$ and $X' \in \mathcal{E}_j\ \}$

Concluding Remarks: Some Questions

- What is the right notion of bisimulation for these models?
- What is the complete logic in a language with the conditional belief/evidence operators? ...in a language with the (un)reliable evidence operator?
- We know that the satisfiability problem is decidable, but what is its complexity?
- What happens when the agent notices an inconsistency in her evidence? (eg., Priority structures, represent the sources)
- ...

Thank you!!