Neighborhood Semantics for Modal Logic
Lecture 3

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Course Plan

✓ **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics

1. **Core Theory**: Relationship with Other Semantics for Modal Logic, Model Theory; Completeness, Decidability, Complexity, Incompleteness

2. **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods

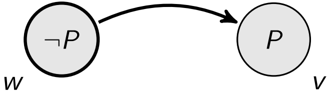
Core Theory

- Neighborhood Semantics in the Broader Logical Landscape
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

The Broader Logical Landscape

- ✓ Relational Models
- ✓ Topological Models
- ✓ *n*-ary Relational Structures
- Plausibility Structures
- First-Order Logic

Beliefs via Plausibility



Epistemic-Plausibility Model: $\mathcal{M} = \langle W, \preceq, V\rangle$

- $w \preceq v$ means w is at least as plausibility as v. ($\preceq$ is reflexive, transitive, connected, well-founded)

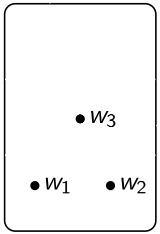
Language: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid B^{\varphi}\psi \mid [\preceq]\varphi \mid A\varphi$

Truth:

- $Max_{\preceq}(X) = \{w \in X \mid \text{there is no } v \in X \text{ such that } w \prec v\}$
- $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$
- $\mathcal{M}, w \models B^{\varphi}\psi$ iff for all $v \in Max_{\preceq}([\![\varphi]\!]_{\mathcal{M}})$, $\mathcal{M}, v \models \psi$

Beliefs via Plausibility

$$W = \{w_1, w_2, w_3\}$$



Beliefs via Plausibility

$W = \{w_1, w_2, w_3\}$

$w_2 \preceq w_1$ and $w_1 \preceq w_2$ (w_1 and w_2 are equi-plausbile)

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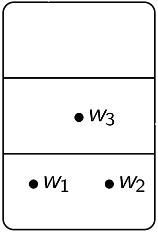

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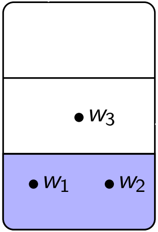
$\{w_1, w_2\} \subseteq Max_{\preceq}(W)$



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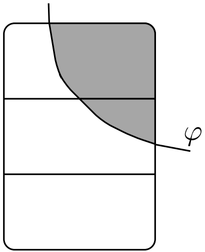
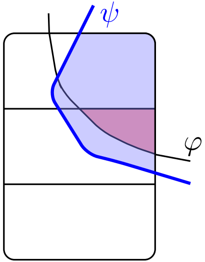


Conditional Belief: $B^{\varphi}\psi$

Beliefs via Plausibility



Conditional Belief: $B^{\varphi}\psi$

$$Max_{\preceq}([\![\varphi]\!]_{\mathcal{M}}) \subseteq [\![\psi]\!]_{\mathcal{M}}$$

Evidence Models and Plausibility Models

What is the precise relationship between evidence models and plausibility models?

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Three issues

1. Plausibility orders that are not *connected*
2. Conditional beliefs on evidence models
3. From evidence to plausibility (and back)

Incomparability in Plausibility Models

In general, we must drop the assumption that $\preceq$ is connected.

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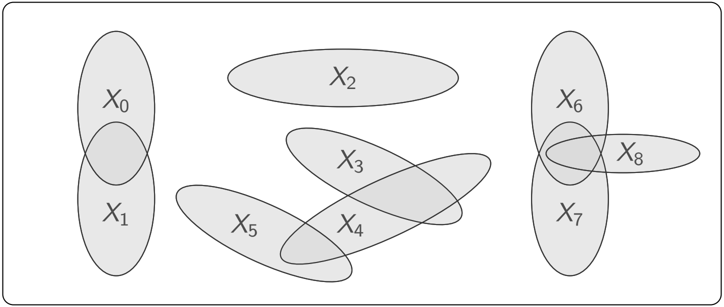
Incomparability arises as the result of receiving incompatible evidence:

Incompatible evidence Compatible evidence

Conditional Beliefs on Evidence Models



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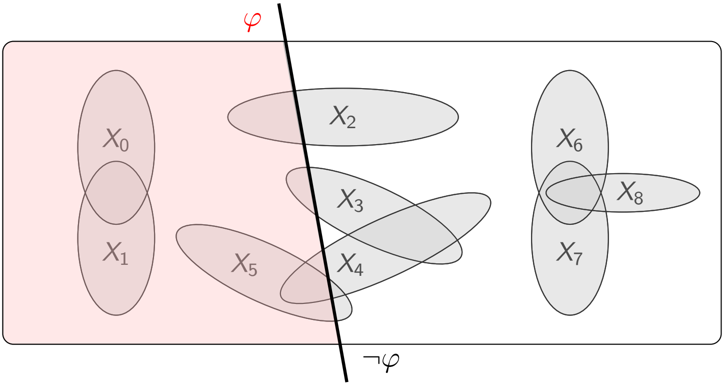

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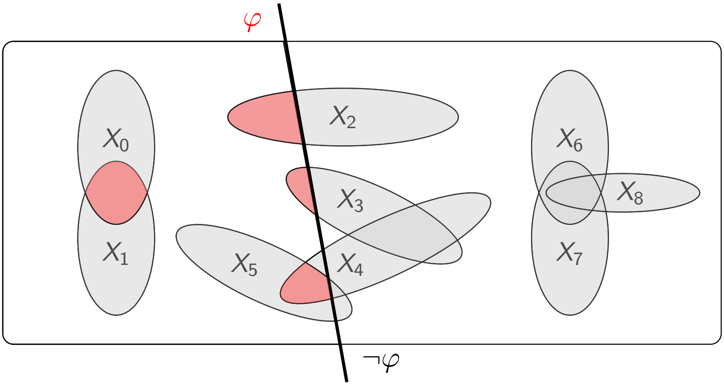

Conditional Beliefs on Evidence Models

$B^\varphi\psi$: "the agent believes ψ conditional on φ."

Main idea: Ignore the evidence that is inconsistent with φ.

Relativized w**-scenario**: Suppose that $X \subseteq W$. Given a collection $\mathcal{X} \subseteq \wp(W)$, let $\mathcal{X}^X = \{Y \cap X \mid Y \in \mathcal{X}\}$. We say that a collection $\mathcal{X}$ of subsets of W has the **finite intersection property relative to** X (X-f.i.p.) if, $\mathcal{X}^X$ as the f.i.p. and is maximal if $\mathcal{X}^X$ is.

- $\mathcal{M}, w \models B^\varphi\psi$ iff for each maximal φ-f.i.p. $\mathcal{X} \subseteq E(w)$, for each $v \in \bigcap \mathcal{X}^\varphi$, $\mathcal{M}, v \models \psi$

Conditional Beliefs: Example

$B\psi \rightarrow B^{\varphi}\psi$ is not valid.

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$B\psi \to B^{\varphi}\psi$ is not valid.

Is $B\psi \to B^{\varphi}\psi \vee B^{\neg\varphi}\psi$ valid?

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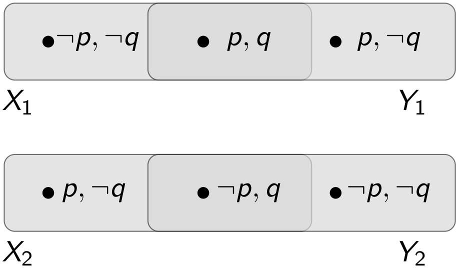

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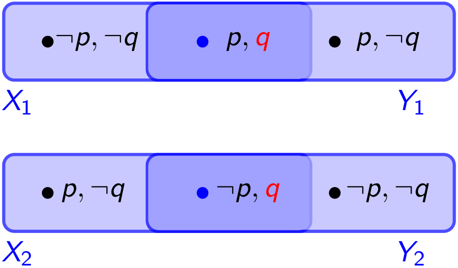

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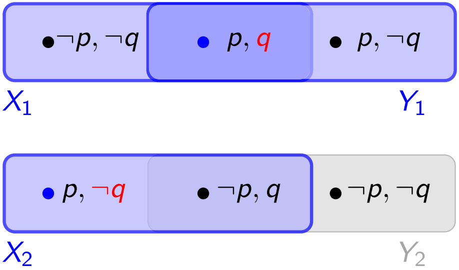

- ✓ $\mathcal{M}, w \models Bq$
- $\mathcal{M}, w \not\models B^{p}q$

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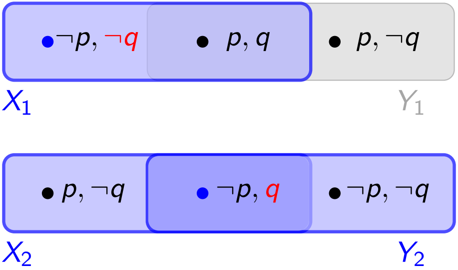

- ✓ $\mathcal{M}, w \models Bq$
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- $\mathcal{M}, w \models \langle\ ]^{\varphi}\psi$ iff there exists an evidence set $X \in E(w)$ consistent with φ such that for all $v \in X \cap [\![\varphi]\!]_{\mathcal{M}}$, $\mathcal{M}, v \models \psi$.

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$\langle\ ]^{\varphi}\psi$ is not equivalent to $\langle\ ](\varphi \rightarrow \psi)$: if there is no evidence consistent with φ, then $\langle\ ]^{\varphi}\psi$ is false.

Plausibility Models $\hookrightarrow$ Evidence Models

Let $\mathcal{M} = \langle W, \preceq, V\rangle$ be a plausibility model.

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- Given a $X \subseteq W$, let $X{\uparrow_{\preceq}} = \{v \in W \mid \exists x \in X \text{ and } x \preceq v\}$
- A set $X \subseteq W$ is $\preceq$-closed if $X{\uparrow} \subseteq X$.

Evidence model generated from $\mathcal{M}$: $EV(\mathcal{M}) = \langle W, \mathcal{E}^{\preceq}, V \rangle$ with $\mathcal{E}^{\preceq} = \{X \mid \emptyset \neq X \text{ is } \preceq\text{-closed }\}$

$\mathcal{L}(\preceq, B, A)$ is generated by

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [B]\psi \mid [\preceq]\varphi \mid [A]\varphi$$

Suppose that $\mathcal{M} = \langle W, \preceq, V\rangle$ is a plausibility model.

- $\mathcal{M}, w \models [B]\varphi$ iff $Max_{\preceq}(W) \subseteq [\![\varphi]\!]_{\mathcal{M}}$
- $\mathcal{M}, w \models [\preceq]\varphi$ iff for all $v \in W$, if $w \preceq v$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

On finite plausibility models, the belief modality $[B]$ is definable in terms of the $[A]$ and $[\preceq]$ modalities:

$$B\varphi \ := \ [A]\langle\preceq\rangle[\preceq]\varphi$$

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The translation $tr_{\preceq} : \mathcal{L}(\langle\;], A) \rightarrow \mathcal{L}([\preceq], A)$ is defined as follows:

- for each $p \in \mathrm{At}$, $tr_{\preceq}(p) = p$;
- $tr_{\preceq}(\neg\varphi) = \neg tr_{\preceq}(\varphi)$ and $tr_{\preceq}(\varphi \wedge \psi) = tr_{\preceq}(\varphi) \wedge tr_{\preceq}(\psi)$;
- $tr_{\preceq}([A]\varphi) = [A](tr_{\preceq}(\varphi))$; and
- $tr_{\preceq}(\langle\;]\varphi) = \langle E\rangle[\preceq](tr_{\preceq}(\varphi))$.

Proposition. Let $\mathcal{M} = \langle W, \preceq, V \rangle$ be a plausibility model. For any $\varphi \in \mathcal{L}(\langle\ ], A)$ and state $w \in W$,

$$\mathcal{M}, w \models tr_{\preceq}(\varphi) \text{ iff } \mathcal{M}^{\preceq}, w \models \varphi.$$

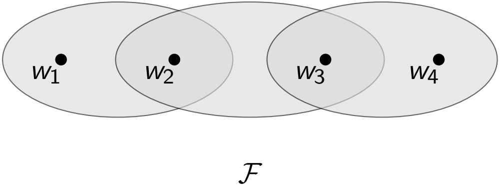
Evidence Models $\hookrightarrow$ Plausibility Models

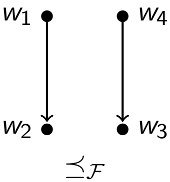
Specialization Order

Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Define $\preceq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \preceq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X$$

Specialization Order: Example





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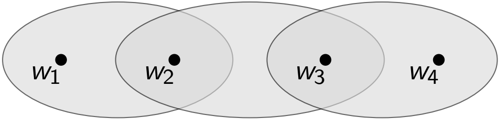
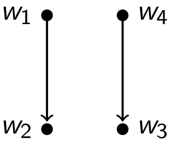
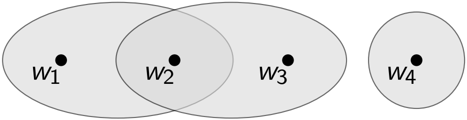


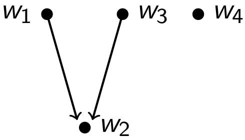
$\mathcal{F}$



$\preceq_{\mathcal{F}}$



$\mathcal{F}'$



$\preceq_{\mathcal{F}'}$

$\mathfrak{M} = \langle W, N, V \rangle$. For each $w \in W$, define a plausibility ordering $\preceq_{N(w)}$.

Taking Stock

Language $\varphi \; := \; p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle \;]\varphi \mid [B]\varphi \mid [A]\varphi \mid [\preceq]\varphi$

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General Model: $\langle W, E, R_B, \preceq, V\rangle$ where

1. for each $w \in W$, $\varnothing \notin E(w)$ and $W \in E(w)$;
2. for all $w, v, u \in W$, if $w \preceq v$ and $w \in X \in E(u)$, then $v \in X$;
3. for all w, v, u, if $w \preceq v$ and $u\, R_B\, v$ then $u\, R_B\, w$.

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Intended Model: $\langle W, E, V\rangle \hookrightarrow \langle W, E, R_B^E, \preceq^E, V\rangle$ where

1. $w\, R_B^E\, v$ iff $v \in \cap\mathcal{X}$ for some w-scenario $\mathcal{X}$
2. $w \preceq^E v$ iff whenever u, X are such that $w \in X \in E(u)$, then $v \in X$

Given an evidence model $\mathfrak{M} = \langle W, E, V\rangle$, define the extended model

$$\mathfrak{M}^{\triangle} = \langle W, E, B_E, \preceq_E, V\rangle.$$

where

- $w\ B_E\ v$ iff $v \in \bigcap \mathcal{X}$ for some w-scenario $\mathcal{X}$, and
- $\preceq_E = \preceq_{\cup_{w\in W} E(w)}$ (i.e., $w \preceq_E v$ iff for any u, X, if $w \in X \in E(u)$, then $v \in X$).

Say that $\mathcal{M}$ is an **intended model** provided $\mathcal{M} = \langle W, E, V\rangle^{\triangle}$

What is the precise relationship between intended models $\mathfrak{M}^{\triangle}$ and extended evidence models $\mathcal{M} = \langle W, E, B, \preceq, V \rangle$.

Lemma. Suppose that $\mathfrak{M} = \langle W, E, V \rangle$ is an evidence model, then $\mathfrak{M}^{\triangle}$ is a model according to the above definition.

Lemma. If $\mathcal{M} = \langle W, \mathcal{E}, B_{\mathcal{E}}, \preceq_{\mathcal{E}}, V\rangle$ is uniform and intended, then for every scenario $\mathcal{X}$ and every $w \in \bigcap \mathcal{X}$, w is $\preceq$-maximal if and only if w lies in $\bigcap \mathcal{X}'$ for some scenario $\mathcal{X}'$.
Moreover, if $\mathcal{M}$ is flat then the sets of the form $\bigcap \mathcal{X}$ with $\mathcal{X}$ a scenario are precisely the $\preceq_{\mathcal{E}}$-equivalence classes of maximal worlds.

The plausibility orders in extended evidence models satisfy an additional property:

Let $\preceq$ be a plausibility order over W. Say $D \subseteq W$ is **directed** if any two elements of D have an upper bound in D.

A plausibility order $\preceq$ satisfies the **boundendess condition** if every directed set D has an upper bound (not necessarily in D).

Proposition. If an evidence model is flat, then its derived plausibility relation satisfies the boundedness condition.

Lemma. If $\mathcal{M}$ is flat and $\preceq_E$ is its derived plausibility relation, then for every w there is v such that $w \preceq_E v$ and v is maximal.

Theorem. Over the class of uniform evidence models with derived plausibility relation, $[A]\langle\preceq\rangle[\preceq]\varphi \rightarrow [B]\varphi$ is valid.

Over the class of models that are moreover flat, the two formulas are equivalent.

Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Define $\preceq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \preceq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X$$

Two ways to generalize:

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2. A set of *reasons* $\mathcal{R} \subseteq \mathcal{F}$ may be associated with arbitrary orderings: $\mathcal{R} \mapsto \preceq_{\mathcal{R}}$.

F. Dietrich and C. List. *Reasons for (prior) belief in Bayesian epistemology*. .

Let $\mathcal{D} \subseteq \wp(W)$ be a set of *doxastic reasons*.

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Let $\mathbb{D}$ be the space of doxastic reasons. Assume that $\mathbb{D}$ is closed under finite intersections and finite unions.

Example

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Possible meeting points (**Doxastic Possibilities**):

Union station (u)
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Doxastic Reasons

$A = \{u\}$: The place in question is where one arrives in DC
$F = \{l, w\}$: The place in question is world-famous
$H = \{w, e\}$: A family lives at the place in question

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$$
\begin{aligned}
\mathcal{D} = \{A, F, H\}: &\quad u \succ_{\mathcal{D}} l \succ_{\mathcal{D}} w \succ_{\mathcal{D}} e \\
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\mathcal{D} = \{H\}: &\quad l \sim_{\mathcal{D}} u \sim_{\mathcal{D}} w \sim_{\mathcal{D}} e \\
\mathcal{D} = \emptyset: &\quad l \sim_{\mathcal{D}} w \sim_{\mathcal{D}} u \sim_{\mathcal{D}} e
\end{aligned}
$$

Can we find a **credibility ordering** $\geq$ on the space of doxastic reasons $\mathbb{D}$ such that

$$w \succeq_{\mathcal{D}} v \text{ iff } \{R \mid w \in R \in \mathcal{D}\} \geq \{R \mid v \in R \in \mathcal{D}\}?$$

Axiom 1 (Principle of insufficient reason): For any $w, v \in W$ and any $\mathcal{D} \in \mathbb{D}$

$$\text{if } \{R \mid w \in R \in \mathcal{D}\} = \{R \mid v \in R \in \mathcal{D}\}, \text{ then } w \sim_{\mathcal{D}} v$$

Axiom 2: For any $w, v \in W$ and any $\mathcal{D}_2, \mathcal{D}_2 \in \mathbb{D}$ with $\mathcal{D}_1 \subseteq \mathcal{D}_2$,

$$\text{if, [for all } R \in \mathcal{D}_2 - \mathcal{D}_1,\ w, v \notin R], \text{ then } [w \succeq_{\mathcal{D}_1} v \Leftrightarrow w \succeq_{\mathcal{D}_2} v]$$

Theorem (Dietrich and List). The agent's plausibility orderings $(\succeq_{\mathcal{D}})_{\mathcal{D} \in \mathbb{D}}$ satisfies Axiom 1 and Axiom 2 if and only if there is a credibility ordering $\geq$ on $\mathbb{D}$ such that for all $\mathcal{D} \in \mathbb{D}$,

$$w \succeq_{\mathcal{D}} v \Longleftrightarrow \{R \mid w \in R \in \mathcal{D}\} \geq \{R \mid v \in R \in \mathcal{D}\}$$

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Axiom 1 (Principle of insufficient reason): For any $w, v \in W$ and any $\mathcal{D} \in \mathbb{D}$

$$\text{if } \{R \mid w \in R \in \mathcal{D}\} = \{R \mid v \in R \in \mathcal{D}\}, \text{ then } w \sim_{\mathcal{D}} v$$

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Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \nvdash K$
- $\mathbf{EM} \nvdash K$
- $\mathbf{EMC} \vdash K$
- $\mathbf{EK} \nvdash M$
- $\mathbf{EK} \nvdash C$

(MP) $$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

(Nec) $$\frac{\varphi}{\Box\varphi}$$

(RE) $$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

(RR) $$\frac{(\varphi_1 \wedge \varphi_2) \rightarrow \psi}{(\Box\varphi_1 \wedge \Box\varphi_2) \rightarrow \Box\psi}$$

(RK) $$\frac{(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi}{(\Box\varphi_1 \wedge \cdots \Box\varphi_n) \rightarrow \Box\psi} \qquad (n \geq 0)$$

Some Notation

- A formula $\varphi \in \mathcal{L}$ is valid in F ($\models_{\mathsf{F}} \varphi$) if for each $\mathbb{F} \in$ F, $\mathbb{F} \models \varphi$.
- We say that a logic **L** is sound with respect to F, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathsf{F}} \varphi$.
- A set of formulas Γ semantically entails φ with respect to F, denoted $\Gamma \models_{\mathsf{F}} \varphi$, if for each $\mathbb{F} \in$ F, if $\mathbb{F} \models \Gamma$ then $\mathbb{F} \models \varphi$.
- A logic **L** is weakly complete with respect to a class of frames F, if $\models_{\mathsf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.
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A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L**-proof set of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Let $\mathbf{L}$ be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally $\mathbf{L}$-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally $\mathbf{L}$-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

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Definition

A neighborhood model $\mathbb{M} = \langle W, N, V \rangle$ is canonical for **L** provided

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- for all $p \in \mathrm{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathfrak{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathfrak{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \not\in P_{\mathbf{L}}\}$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathfrak{M}$ *is canonical for* $\mathbf{L}$,

$$[\![\varphi]\!]_{\mathfrak{M}} = |\varphi|_{\mathbf{L}}$$

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The Proofs

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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Lemma
If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem
The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N_{\mathbf{EM}}^{min} \rangle$ is not closed under supersets.

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Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}} \rangle$ is not closed under supersets.

Lemma

Suppose that $\mathfrak{M} = sup(\mathfrak{M}^{min}_{\mathbf{EM}})$. Then $\mathfrak{M}$ is canonical for **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem
The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem
The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

The Normal Situation

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

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Theorem: K is sound and strongly complete with respect to the class of all Kripke frames.

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Theorem: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

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Theorem: **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive Kripke frames.

Incompleteness

There are (consistent) modal logics that are incomplete:

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Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \to \varphi$
- $\Box\Diamond\varphi \to \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \to \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)) \to \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

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There is no class of frames validating precisely the formulas in **TMEQ**.

J. van Benthem. *Two Simple Incomplete Modal Logics*. Theoria (1978).

BAO

Definition A boolean algebra with operators is a pair $\mathfrak{B} = \langle \mathfrak{A}, m \rangle$ where $\mathfrak{A}$ is a bolean algebra and m is a unary operator on $\mathfrak{A}$ such that:

- $m(x + y) = m(x) + m(y)$
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Example: Given a Kripke frame $\mathbb{F} = \langle W, R \rangle$, let $\mathfrak{A} = \langle \wp(W), \cap, \cup, \cdot^C \rangle$ and $m : \wp(X) \to \wp(X)$ is defined as:

$$m(X) = \{y \in W \mid \exists x \in X \text{ such that } yRx\}$$

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Theorem *Any* normal modal logic is complete with respect to some class of boolean algebras with operators.

General Frames

Definition A general frame is a pair $\langle \mathbb{F}, \mathcal{A} \rangle$ where $\mathbb{F} = \langle W, R \rangle$ is a Kripke frame, and $\emptyset \neq \mathcal{A} \subseteq \wp(W)$ is a collection of admissible sets closed under the following operations:

- union: if $X, Y \in \mathcal{A}$ then $X \cup Y \in \mathcal{A}$
- relative complement: if $X \in \mathcal{A}$ then $W - X \in \mathcal{A}$
- modal operations: if $X \in \mathcal{A}$ then $m(X) \in \mathcal{A}$

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Theorem *Any* normal modal logic **L** is sound and strongly complete with respect to some class of general frames.

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

Incompleteness

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There are two logics $\mathbf{L}$ and $\mathbf{L}'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for $\mathbf{L}$ and $\mathbf{L}'$ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

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There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L$'$ is above **S4** (adapts Fine's incomplete logic)

$$
\begin{aligned}
A_i &= \Box(q_i \to r) \qquad (i = 1, 2)\\
B_i &= \Box(r \to \Diamond q_i) \qquad (i = 1, 2)\\
C_1 &= \Box\neg(q_1 \wedge q_2)\\
A &= r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge\\
&\quad C_1 \to \Diamond(r \wedge \Box(r \to (q_1 \vee q_2))\\
D &= (p \wedge \Diamond\Diamond q) \to (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p))\\
E &= (\Box p \wedge \neg\Box\Box p) \to \Diamond(\Box\Box p \wedge \neg\Box\Box\Box p)\\
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Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

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Theorem. (Gerson) The formula F is valid in all neighborhood frames for **L**, but it is not provable in **L**.

Comparing Relational and Neighborhood Semantics

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

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What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

The general situation is not very well understood.

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Notable exceptions:

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Recovering Completeness

Definition

A general neighborhood frame is a tuple $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$, where $\langle W, N \rangle$ is a neighborhood frame and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator.

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A valuation $V : \text{At} \rightarrow \wp(W)$ is admissible for a general frame $\langle W, N, \mathcal{A} \rangle$ if for each $p \in \text{At}$, $V(p) \in \mathcal{A}$.

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Definition

Suppose that $\mathfrak{F}^g = \langle W, N, \mathcal{A} \rangle$ is a general neighborhood frame. A general modal based on $\mathfrak{F}^g$ is a tuple $\mathfrak{M}^g = \langle W, N, \mathcal{A}, V \rangle$ where V is an admissible valuation.

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Lemma

Let $\mathfrak{M}^g$ be an general neighborhood model. Then for each $\varphi \in \mathcal{L}$, $[\![\varphi]\!]_{\mathfrak{M}^g} \in \mathcal{A}$.

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Lemma

Let **L** *be any logic extending* **E**. *Then the general canonical frame validates* **L** *(*$\mathfrak{F}^g_{\mathbf{L}} \models \mathbf{L}$*).*

Corollary

Any modal logic extending **E** *is strongly complete with respect to some class of general frames.*

Summary

For any modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- If **L** is neighborhood complete, then it is algebraically complete
- **L** is complete with respect to its class of general frames

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There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness

End of lecture 3