Neighborhood Semantics for Modal Logic
Lecture 2

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Plan

- ✓ Introductory Remarks
- ✓ Background: Relational Semantics for Modal Logic
- ✓ Why *Non-Normal* Modal Logic?
- Fundamentals
  - ✓ Subset Spaces
  - Neighborhood Semantics
- Why Neighborhood Semantics?

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N\rangle$ is a tuple $\langle W, N, V\rangle$ where $V : \mathsf{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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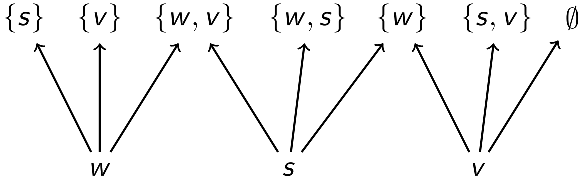

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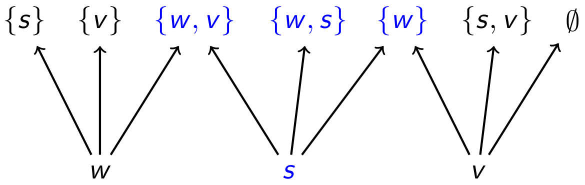

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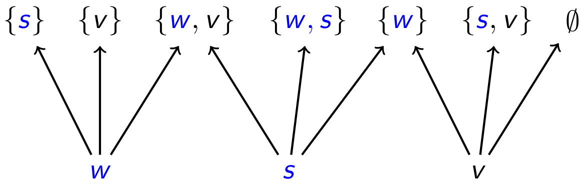

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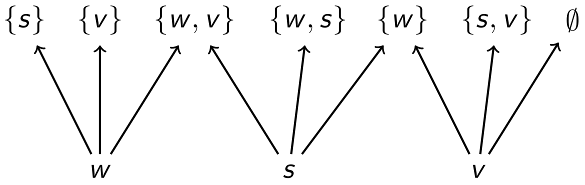

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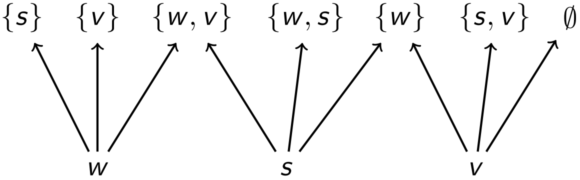

$$\mathfrak{M}, s \models \Box p$$

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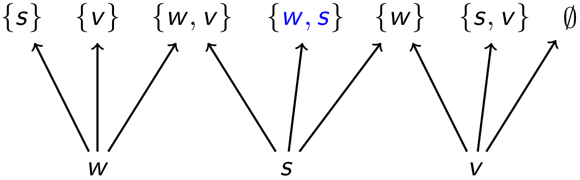

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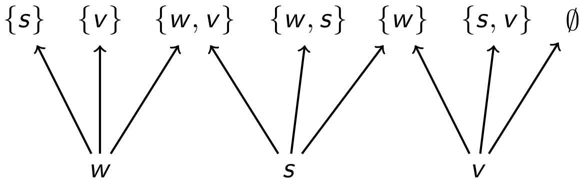

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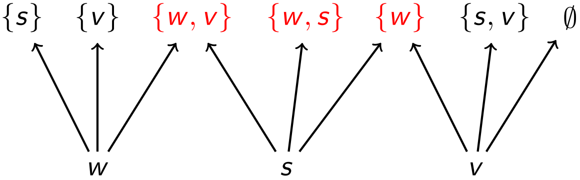

$$\mathfrak{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

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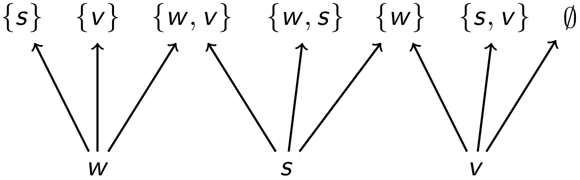

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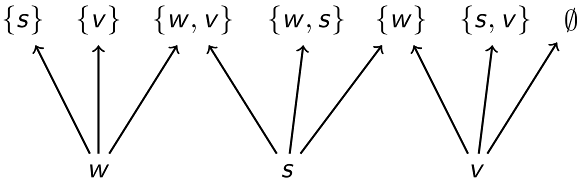

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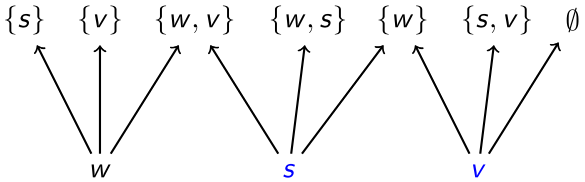

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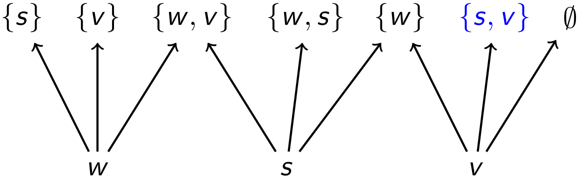

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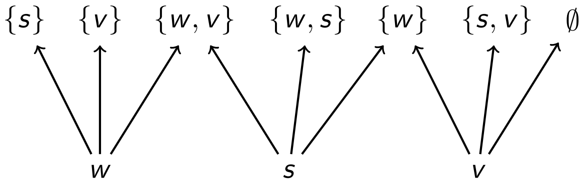

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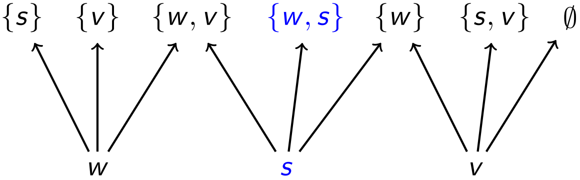

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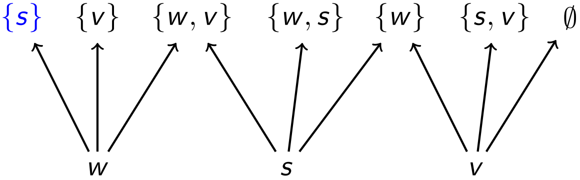

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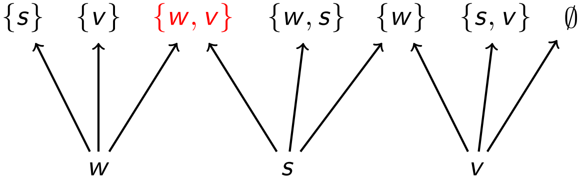

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Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
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Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

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Lemma

1. *If $\varphi \to \psi$ is valid in $\mathfrak{M}$, then so is $\langle\,]\varphi \to \langle\,]\psi$.*
2. $\langle\,](\varphi \wedge \psi) \to (\langle\,]\varphi \wedge \langle\,]\psi)$ *is valid in $\mathfrak{M}$*

Investigate analogous results for the other modal operators defined above.

Two routes to a logical framework

✓ Identify interesting patterns that you (do not) want to represent

2. Identify interesting structures that you want to reason about

A (Dynamic) Logic of Knowledge, Evidence and Belief

J. van Benthem and EP. *Dynamic Logics of Evidence-Based Beliefs*. Studia Logica, 99, pp. 61 - 92, 2011.

J. van Benthem, D. Fernández-Duque and EP. *Evidence Logic: A New Look at Neighborhood Structures*. Proceedings of Advances in Modal Logic, King's College Publications, 2012.

J. van Benthem, D. Fernández-Duque and EP. *Evidence and Plausibility in Neighborhood Structures*. Annals of Pure and Applied Logic, 2013.

Setting the Stage: Evidence

- Dempster-Shafer Theory of Evidence

G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

- Bayesian Confirmation Theory (eg., E confirms H iff $p(H \mid E) > p(H)$)

B. Fitelson. *The Plurality of Bayesian Measures of Confirmation and the Problem of Measure Sensitivity*. Philosophy of Science 66, 1999.

Setting the Stage: Evidence

- Artemov/Fitting's Justification Logic ($t\!:\!\varphi$: "t is a *justification*/*proof* for φ")

S. Artemov and M. Fitting. *Justification logic*. The Stanford Encyclopedia of Philosophy, 2012.

- Moss and Parikh's "topologic" ($x, U \models \varphi$: "φ is true at the state x given that the current *evidence*/ *"measurement"* gathered is U")

L. Moss and R. Parikh. *Topological reasoning and the logic of knowledge*. Proceedings of TARK IV, Morgan Kaufmann, 1992.

Setting the Stage: Reasons

- Kratzer Semantics (modal base), believing for a *reason* (deriving an ordering on worlds from an ordering over propositions)

A. Kratzer. *What* must *and* can *must and can mean*. Linguistics and Philosophy 1 (1977) 337355.

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C. List and F. Dietrich. *Reasons for (prior) belief in bayesian epistemology*. 2012.

- Reason management (Default logic with priorities)

J. Horty. *Reasons as Defaults*. 2012.

Modeling Evidence: Some Distinctions

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In between: family of subsets representing evidence from received from various (possible unreliable) sources

Evidence Models: Basic Assumptions

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3. Despite the fact that sources may not be reliable or jointly inconsistent, they are all the agent has for forming beliefs.

Evidential States

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In addition, much of the literature would suggest a 'monotonicity' assumption:

> *If the agent has evidence* X *and* $X \subseteq Y$ *then the agent has evidence* Y*.*

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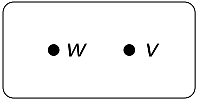
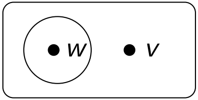
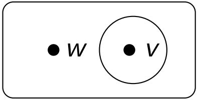


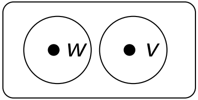
There is no evidence for or against p.



There is evidence that supports p.



There is evidence that rejects p.



There is evidence that supports p and also evidence that rejects p.

Evidence Models

Evidence model: $\mathcal{M} = \langle W, E, V \rangle$

- W is a non-empty set of worlds,
- $V : \mathsf{At} \to \wp(W)$ is a valuation function, and
- $E : W \to \wp(\wp(W))$ is an evidence relation

  $X \in E(w)$: "the agent accepts X as evidence at state w".

Uniform evidence model (E is a constant function): $\langle W, \mathcal{E}, V \rangle, w$ where $\mathcal{E}$ is the fixed family of subsets of W related to each state by E.

Assumptions

(Cons) For each state w, $\emptyset \not\in E(w)$.

(Triv) For each state w, $W \in E(w)$.

The Basic Language $\mathcal{L}$ of Evidence and Belief

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\,]\varphi \mid [B]\varphi \mid [A]\varphi$$

- $\langle\,]\varphi$ says that "the agent has evidence that φ is true" (i.e., "the agent has evidence for φ")
- $[B]\varphi$ says that "the agents believes that φ is true" (based on her evidence)
- $[A]\varphi$ says that "φ is true in all states" (which we interpret as the agent's *knowledge*)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p) \qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
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- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

“Having evidence for φ” vs. “Accepting φ as evidence”

We do not assume that the evidence sets are closed under supersets, though our semantic definition implies that the set of propositions that the agent has *evidence for* is closed under weakening.

So, an agent can have *evidence for* X without *accepting* the set X as evidence.

Defining Beliefs

w-**scenario**: A maximal family of evidence sets $\mathcal{X} \subseteq E(w)$ that has the **finite intersection property** (f.i.p.: for each finite subfamily $\{X_1, \ldots, X_n\} \subseteq \mathcal{X}$, $\bigcap_{1 \leq i \leq n} X_i \neq \emptyset$).

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An agent believes φ at w if each w-scenario implies that φ is true (i.e., φ is true at each point in the intersection of each w-scenario).

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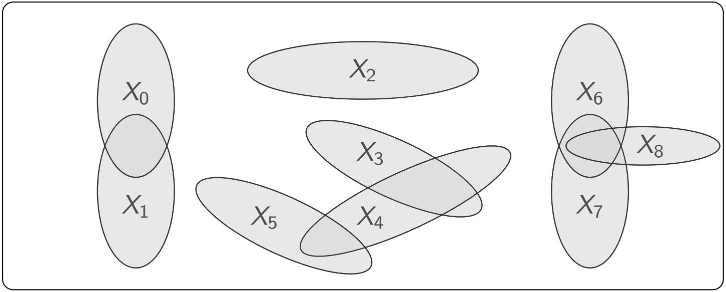
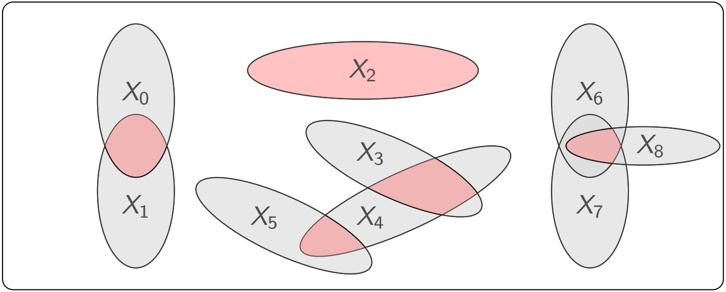


Defining Beliefs



Our definition of belief is very conservative, many other definitions are possible (there exists a w-scenario, "most" of the w-scenarios,...)

Truth

- $\mathcal{M}, w \models p$ iff $w \in V(p) \qquad (p \in \mathsf{At})$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \langle\ ]\varphi$ iff there exists X such that wEX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$

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- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models [B]\varphi$ for all w-scenarios $\mathcal{X} \subseteq E(w)$, for all $v \in \bigcap \mathcal{X}$, $\mathcal{M}, v \models \varphi$

Notation for the truth set: $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$

Flat Evidence Models

An evidence model $\mathcal{M}$ is **flat** if every scenario on $\mathcal{M}$ has non-empty intersection.

Proposition. The formula $\langle\ ]\varphi \rightarrow \langle B\rangle\varphi$ is valid on the class of flat evidence models, but not on the class of all evidence models.

1. Prove that $\langle\ ]\varphi \wedge [A]\psi \leftrightarrow \langle\ ](\varphi \wedge [A]\psi)$ is valid on all evidence models.
2. Prove that $[B]\varphi \rightarrow [A][B]\varphi$ is valid on all uniform evidence models.
3. Show that $\langle\ ]\varphi \rightarrow \langle\ ]\langle\ ]\varphi$ is only valid on uniform evidence models.

Coalitional Logic

M. Pauly. *A Modal Logic for Coalitional Powers in Games*. Journal of Logic and Computation, 12:1, pp. 149 - 166, 2002.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Strategic Game Forms

$$\langle N, \{S_i\}_{i \in N}, O, o \rangle$$

- N is a finite set of players;
- for each $i \in N$, S_i is a non-empty set (elements of which are called actions or strategies);
- O is a non-empty set (elements of which are called **outcomes**); and
- $o : \Pi_{i \in N} S_i \rightarrow O$ is a function assigning an outcome

		Bob	
		t_1	t_2
Ann	s_1	o_1	o_2
	s_2	o_2	o_3
	s_3	o_4	o_1

α-Effectivity

$S = \Pi_{i \in N} S_i$ are called **strategy profiles**. Given a strategy profile $s \in S$, let s_i denote i's component and s_{-i} the profile of strategies from s for all players except i.

A strategy for a coalition C is a sequence of strategies for each player in C, i.e., $s_C \in \Pi_{i \in C} S_i$ (similarly for $s_{\overline{C}}$, where $\overline{C}$ is $N - C$).

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Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o \rangle$ be a strategic game form. An α-**effectivity function** is a map $E_G^\alpha : \wp(N) \to \wp(\wp(O))$ defined as follows: For all $C \subseteq N$, $X \in E_G^\alpha(C)$ iff there exists a strategy profile s_C such that for all $s_{\overline{C}} \in \Pi_{i \in N-C} S_i$, $o(s_C, s_{\overline{C}}) \in X$.

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

α-Effectivity vs. β-Effectivity

$\exists$ "something a player/a coalition *can* do" such that $\forall$ "actions of the other players/nature"...

$\forall$ "(joint) actions of the other players", $\exists$ "something the agent/coalition can do"...

Bob

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Bob

Ann

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s_2	o_2	o_3
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$$E^{\alpha}_{G_0}(\{A\}) = sup(\{\{o_1, o_2\}, \{o_2, o_3\}, \{o_1, o_4\}\})$$

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$$E^{\alpha}_{G_0}(\emptyset) = \{\{o_1, o_2, o_3, o_4, o_5, o_6\}\}$$

Playable Effectivity Functions

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$E(\{i\}) = \{X \mid X \subseteq \mathbb{N} \text{ is infinite}\}$;
$E(\emptyset) = \{X \mid X \subseteq \mathbb{N} \text{ is cofinite (i.e., } \overline{X} \text{ is finite)}\}$;

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Claim. E satisfies Liveness, Safety, N-maximality, Outcome Monotonicity, Superadditivity, but is not the effectivity function of any game.

Core-Complete

Suppose that $(W, \mathcal{F})$ is a monotonic subset space. The **non-monotonic core**, denoted $\mathcal{F}^{nc}$, is a subset of $\mathcal{F}$ defined as follows:

$\mathcal{F}^{nc} = \{X \mid X \in \mathcal{F}$ and for all $X' \subseteq W$, if $X' \subsetneq X$, then $X' \notin \mathcal{F}\}$.

Does every subset space $(W, \mathcal{F})$ have a non-monotonic core?

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Does every subset space $(W, \mathcal{F})$ have a non-monotonic core? No.

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Does every subset space $(W, \mathcal{F})$ have a non-monotonic core? No.

A monotonic collection of sets $\mathcal{F}$ is **core-complete** provided for all $X \in \mathcal{F}$, there exists a $Y \in \mathcal{F}^{nc}$ such that $Y \subseteq X$.

Observation. Suppose that $G = \langle N, \{S_i\}_{i \in N}, O, o\rangle$ is a strategic game form and E_G^α is the associated α-effectivity function. Then the non-monotonic core of $E_G^\alpha(\emptyset) = \{range(o)\}$, where $range(o) = \{x \in O \mid \text{there is a } s \in \Pi_{i \in N} S_i \text{ such that } o(s) = x\}$.

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Claim. If $E(\emptyset) = \{Y \mid Y \text{ is co-finite}\}$, then $E^{nc}(\emptyset) = \emptyset$.

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Claim. If $E(\emptyset) = \{Y \mid Y \text{ is co-finite}\}$, then $E^{nc}(\emptyset) = \emptyset$.

6. (*Empty Coalition*) $E(\emptyset)$ is core complete.

Characterizing Playable Effectivity Functions

Theorem (Pauly 2001; Goranko, Jamorga and Turrini 2013). If $E : \wp(N) \rightarrow \wp(\wp(O))$ is a function that satisfies the conditions 1-6 given above, then $E = E^{\alpha}_{G}$ for some strategic game form.

V. Goranko, W. Jamroga, and P. Turrini. *Strategic Games and Truly Playable Effectivity Functions*. Journal of Autonomous Agents and Multiagent Systems, 26(2), pgs. 288 - 314, 2013.

M. Pauly. *Logic for Social Software*. PhD Thesis, Institute for Logic, Language and Computation, 2001.

Coalitional Models

A coalitional logic model is a tuple $\mathcal{M} = \langle W, E, V \rangle$ where W is a set of states, $E : W \to (\wp(N) \to \wp(\wp(W)))$ assigns to each state a playable effectivity function, and $V : \mathsf{At} \to \wp(W)$ is a valuation function.

$$\mathcal{M}, w \models [C]\varphi \text{ iff } [\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\} \in E(w)(C)$$

Coalitional Logic: Axiomatics

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Subset Space Models

L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. TARK (1992).

Subset Models

A **Subset Frame** is a pair $\langle W, \mathcal{O}\rangle$ where

- W is a set of states
- $\mathcal{O} \subseteq \wp(W)$ is a set of subsets of W, i.e., a set of *observations*

Neighborhood Situation: Given a subset frame $\langle W, \mathcal{O}\rangle$, (w, U) is called a neighborhood situation, provided $w \in U$ and $U \in \mathcal{O}$.

Model: $\langle W, \mathcal{O}, V\rangle$, where $V : \mathrm{At} \to \wp(W)$ is a valuation function.

Language: $\varphi := p \mid \varphi \wedge \varphi \mid \neg\varphi \mid K\varphi \mid \Diamond\varphi$.

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- $w, U \models \Diamond\varphi$ iff there is a $V \in \mathcal{O}$ such that $w \in V$ and $w, V \models \varphi$

Axioms

1. All propositional tautologies
2. $(p \to \Box p) \wedge (\neg p \to \Box \neg p)$, for $p \in \mathsf{At}$.
3. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
4. $\Box\varphi \to \varphi$
5. $\Box\varphi \to \Box\Box\varphi$
6. $K(\varphi \to \psi) \to (K\varphi \to K\psi)$
7. $K\varphi \to \varphi$
8. $K\varphi \to KK\varphi$
9. $\neg K\varphi \to K\neg K\varphi$
10. $K\Box\varphi \to \Box K\varphi$

We include the following rules: modus ponens, K_i-necessitation and $\Box$-necessitation.

Theorem

The previous axioms are sound and complete for the class of all subset models.

L. Moss and R. Parikh. *Topological Reasoning and The Logic of Knowledge*. TARK (1992).

Fact: $\Box\Diamond\varphi \to \Diamond\Box\varphi$ is sound for spaces closed under intersections.

Fact: $\Diamond\varphi \wedge L\Diamond\psi \to \Diamond[\Diamond\varphi \wedge L\Diamond\psi \wedge K\Diamond L(\varphi \vee \psi)]$ is sound for spaces closed under binary unions.

Overview of Results

- (Georgatos: 1993, 1994, 1997) completely axiomatized Topologic where $\mathcal{O}$ is restricted to a topology and showed that the logic has the finite model property. Similarly for treelike spaces.

- (Weiss and Parikh: 2002) showed that an infinite number of axiom schemes is required to axiomatize Topologics in which $\mathcal{O}$ is closed under intersection.

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Plan

- ✓ Introductory Remarks
- ✓ Background: Relational Semantics for Modal Logic
- ✓ Why *Non-Normal* Modal Logic?
- ✓ Fundamentals
  - ✓ Subset Spaces
  - ✓ Neighborhood Semantics
- ✓ Why Neighborhood Semantics?

Plan

- Neighborhood Semantics in the Broader Logical Landscape
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

The Broader Logical Landscape

- Relational Models
- Topological Models
- n-ary Relational Structures
- Plausibility Structures
- First-Order Logic

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

From Kripke Frames to Neighborhood Frames

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is R-necessary at w if $R^{\rightarrow}(w) \subseteq X$.

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}^R_w$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$

- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let* $\langle W, R\rangle$ *be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let* $\langle W, N\rangle$ *be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

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Theorem

✓ *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. □

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

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A neighborhood of w is any set X such that there is an $O \in \mathcal{T}$ with $w \in O \subseteq N$

Let $\mathcal{T}_w$ be the collection of all neighborhoods of w.

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Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.

Topological Models for Modal Logic

The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X \subseteq W$. Then

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Topological Models for Modal Logic

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Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\bot \leftrightarrow \bot, \Box\top \leftrightarrow \top$
3. $\Box\varphi \rightarrow \varphi$
4. $\Box\Box\varphi \leftrightarrow \Box\varphi$
5. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V \rangle$ where $\langle W, \mathcal{T} \rangle$ is a topological space and V a valuation function.

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V \rangle$ where $\langle W, \mathcal{T} \rangle$ is a topological space and V a valuation function.

$$\mathbb{M}^T, w \models \Box\varphi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \varphi$$

$$(\Box\varphi)^{\mathbb{M}^T} = Int((\varphi)^{\mathbb{M}^T})$$

From Neighborhoods to Topologies

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A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

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Fact: A family $\mathcal{B}$ of subsets of W is a basis for some topology if

- for each $w \in W$ there is a $U \in \mathcal{B}$ such that $w \in U$
- for each $U, V \in \mathcal{B}$, if $w \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $w \in W \subseteq U \cap V$

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

Then there is a topological model that is point-wise equivalent to $\mathbb{M}$.

J. van Benthem and G. Bezhanishvili. *Modal Logics of Space*. Handbook of Spatial Logics, pgs. 217 - 298, 2007.

Generalized Relational Models

- n-ary relations
- multiple relations
- non-normal worlds

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

n-ary Relations

$$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$$

An n-**ary relational model** is a tuple $\langle W, R, V\rangle$ where W is a non-empty set and $R \subseteq W^n$ is an n-ary relation ($R \subseteq W^n$) and $V : \mathsf{At} \rightarrow \wp(W)$ is a valuation function. (Assume $n \geq 2$)

n-ary Relations

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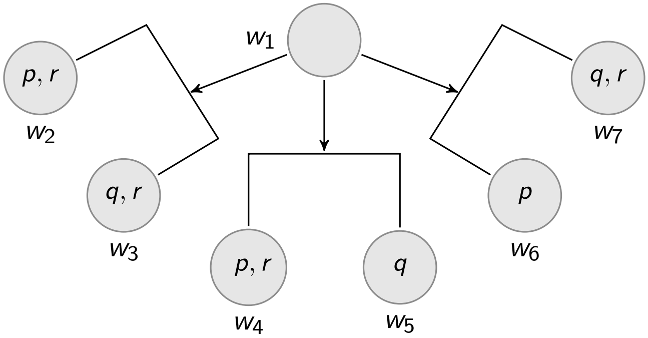
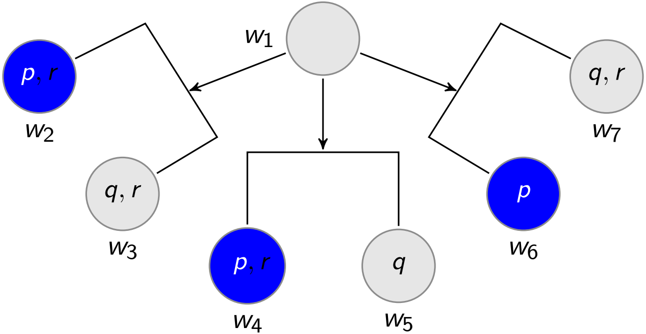
- $\mathcal{M}^n, w \models \Box\varphi$ iff for all $(w_1, \ldots, w_{n-1}) \in W^{n-1}$, if $(w, w_1, \ldots, w_n) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathcal{M}^n, w_i \models \varphi$.

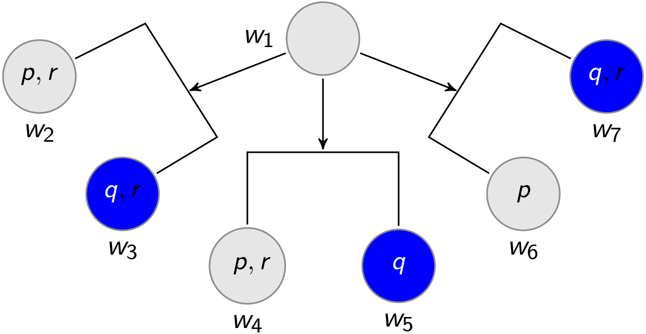
- $\mathcal{M}^n, w \models \Diamond\varphi$ iff there exists $(w_1, \ldots, w_n) \in W^{n-2}$ such that $(w, w_1, \ldots, w_n) \in R$, and for all i such that $1 \leq i \leq n$, we have $\mathcal{M}^n, w_i \models \varphi$.



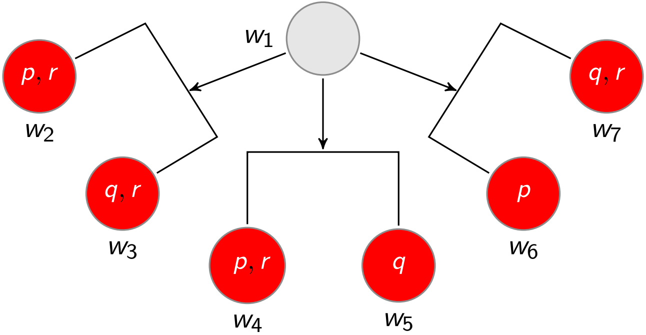
w_1
p, r
w_2
q, r
w_3
p, r
w_4
q
w_5
p
w_6
q, r
w_7



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)



- $\mathcal{M}^3, w_1 \models \Box p$ (and $\mathcal{M}^3, w_1 \models \Box\neg p$)
- $\mathcal{M}^3, w_1 \models \Box q$ (and $\mathcal{M}^3, w_1 \models \Box\neg q$)



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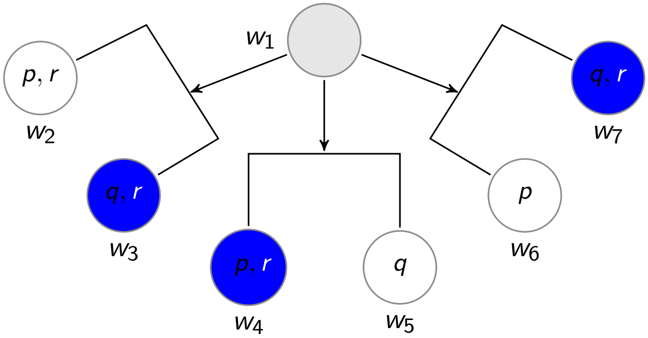

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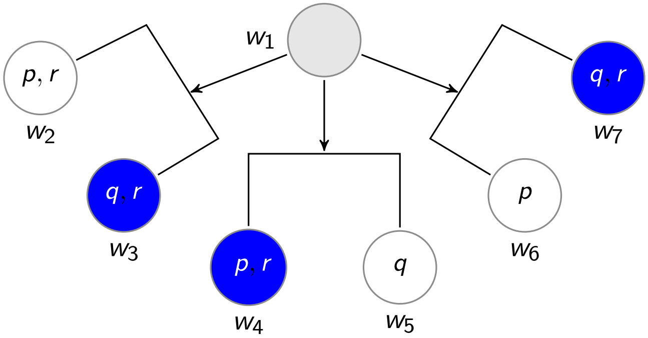

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- $\mathcal{M}^3, w_1 \not\models \Box(p \wedge q)$
- $\mathcal{M}^3, w_1 \models \Box r$
- $\mathcal{M}^3, w_1 \models \Box((p \wedge r) \vee (q \wedge r))$

$$(C^n) \qquad \bigwedge_{i=1}^{n} \Box\varphi_i \rightarrow \Box \bigvee_{1\leq k,l\leq n, k\neq l} (\varphi_k \wedge \varphi_l)$$

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Example:

$(\Box\varphi_1 \wedge \Box\varphi_2 \wedge \Box\varphi_3) \rightarrow \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3))$

Suppose that $\mathbf{L}(\mathfrak{C}^n) = \{\varphi \in \mathcal{L}(\mathsf{At}) \mid \text{for all } \mathcal{F}^n \in \mathfrak{C}^n, \mathcal{F}^n \models \varphi\}$.

$$\mathbf{EMN} = \bigcap_{n \geq 2} \mathbf{L}(\mathfrak{C}^n)$$

Theorem. The logic $\mathbf{EMNC}^n$ is sound and complete for the class $\mathfrak{C}^n$ of n-ary relational frames.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is finite monotonic neighborhood model such that for all $w \in W$, $N(w) \neq \emptyset$. Then, there is an n-ary relational model $\mathcal{M}^N = \langle W^N, R^N, V^N \rangle$ that is modally equivalent to $\mathcal{M}$.

Proposition. Suppose that $\mathcal{M}^n = \langle W, R, V \rangle$ is a finite n-ary relational model. Then, there is a finite monotonic neighborhood model $\mathcal{M}^R = \langle W^R, N^R, V^R \rangle$ that is modally equivalent to $\mathcal{M}^n$.

Multi-Relational Semantics/Non-Normal Modal Logics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

Multi-Relational Semantics/Non-Normal Modal Logics

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$\mathbb{M}, w \models \Box\varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

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Are multi-relational semantics *equivalent* to neighborhood semantics?

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Are multi-relational semantics *equivalent* to neighborhood semantics? Almost

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A world is called impossible if nothing is necessary and everything is possible.

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A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

Multi-Relational Semantics/Non-Normal Modal Logics

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A multi-relational model with impossible worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \notin Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Multi-Relational Semantics/Non-Normal Modal Logics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Synthese Library, 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic, 5(2), pgs. 113-134, 2000.

G. Governatori and A. Rotolo. *On the axiomatization of Elgesem's logic of agency and ability*. Journal of Philosophical Logic, 34(4), pgs. 403 - 431, 2005.

Let $Th_{\mathcal{L}}(\mathcal{M}, w) = \{\varphi \in \mathcal{L} \mid \mathcal{M}, w \models \varphi\}$

Suppose that M and M′ are two classes of models for $\mathcal{L}$. Say that $\mathcal{M}, w$ is $\mathcal{L}$-equivalent to $\mathcal{M}', w'$, denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, provided $Th_{\mathcal{L}}(\mathcal{M}, w) = Th_{\mathcal{L}}(\mathcal{M}', w')$.

A class of models M is $\mathcal{L}$-equivalent to a class of models M′ provided for each pointed model $\mathcal{M}, w$ from M, there exists a pointed model $\mathcal{M}', w'$ from M′ such that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, and *vice versa*.

- The class $\mathsf{K} = \{\mathcal{M} \mid \mathcal{M}$ is a relational model $\}$ is modally equivalent to the class $\mathsf{M}_{aug} = \{\mathcal{M} \mid \mathcal{M}$ is an augmented neighborhood model$\}$

- The class $\mathsf{K}^n = \{\mathcal{M}^n \mid \mathcal{M}$ is an n-ary relational model $\}$ is modally equivalent to the class $\mathsf{M}_{reg} = \{\mathcal{M} \mid \mathcal{M}$ is a consistent regular neighborhood model$\}$

- The class $\mathsf{T} = \{\mathcal{M}^T \mid \mathcal{M}$ is a topological model $\}$ is modally equivalent to the class $\mathsf{M}_{\mathsf{S4}} = \{\mathcal{M} \mid \mathcal{M}$ is an **S4** neighborhood model$\}$

End of lecture 2