Neighborhood Semantics for Modal Logic
Lecture 1

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August 11, 2014

Course Plan

1. **Introduction and Motivation**: Background (Relational Semantics for Modal Logic), Subset Spaces, Neighborhood Structures, Motivating Non-Normal Modal Logics/Neighborhood Semantics
   (Monday, Tuesday)
2. **Core Theory**: Completeness, Decidability, Complexity, Incompleteness, Relationship with Other Semantics for Modal Logic, Model Theory
   (Tuesday, Wednesday, Thursday)
3. **Extensions and Applications**: First-Order Modal Logic, Common Knowledge/Belief, Dynamics with Neighborhoods: Game Logic and Game Algebra, Dynamics on Neighborhoods
   (Thursday, Friday)

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Course Website: `pacuit.org/esslli2014/nbhd`

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Course notes: *Neighborhood Semantics for Modal Logic*, by EP, available on the website (updated during the week)

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Additional readings

- *Modal Logic: an Introduction*, Chapters 7 - 9, by Brian Chellas
- *Monotonic Modal Logics* by Helle Hvid Hansen, available at `www.few.vu.nl/~hhansen/papers/scriptie_pic.pdf`
- *Modal Logic* by P. Blackburn, M. de Rijke and Y. Venema.

Plan

- Introductory Remarks
- Background: Relational Semantics for Modal Logic
- Why *Non-Normal* Modal Logic?
- Fundamentals
    - Subset Spaces
    - Neighborhood Semantics
- Why Neighborhood Semantics?

The Basic Modal Language: $\mathcal{L}$

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p is an atomic proposition (Let At be the set of atomic propositions)

One Language, Many Interpretations

tense: henceforth, eventually, previously, now, tomorrow, yesterday, since, until, it will have been, it is being,...

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metalogic: it is valid/satisfiable/provable/consistent that

game/action: there exist a strategy/action to guarantee that

Relational Structures

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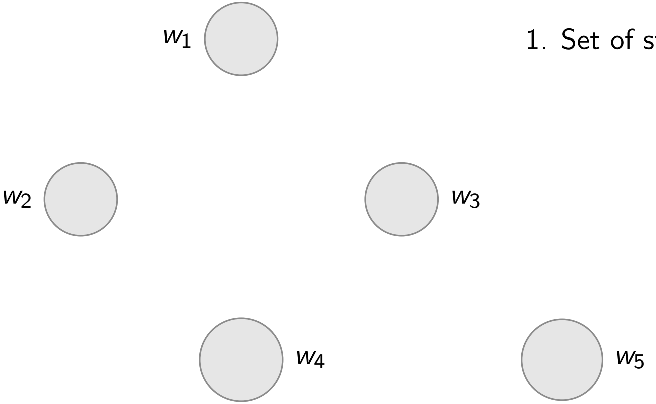
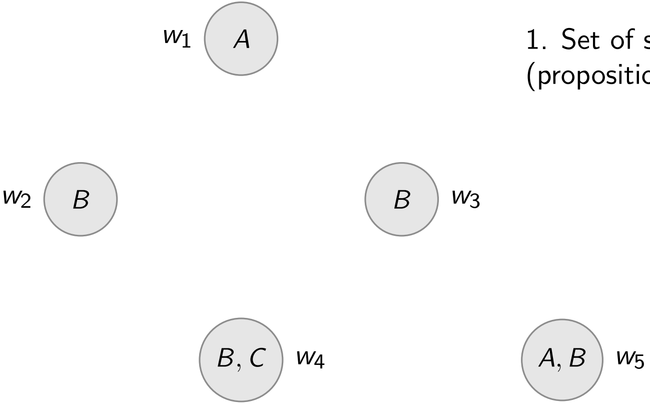
Relational (Kripke) Model: $\langle W, R, V \rangle$

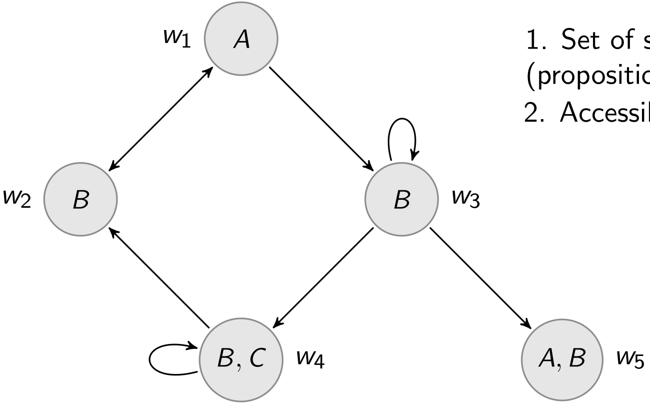
- $\langle W, R \rangle$ is a frame
- $V : \mathsf{At} \to \wp(W)$



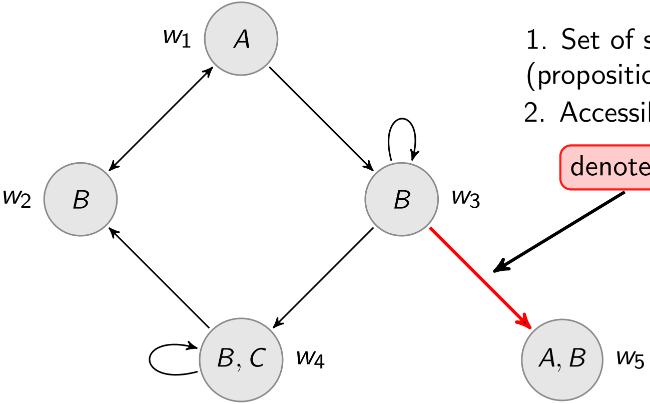
1. Set of states



1. Set of states (propositional valuations)



1. Set of states (propositional valuations)
2. Accessibility relation



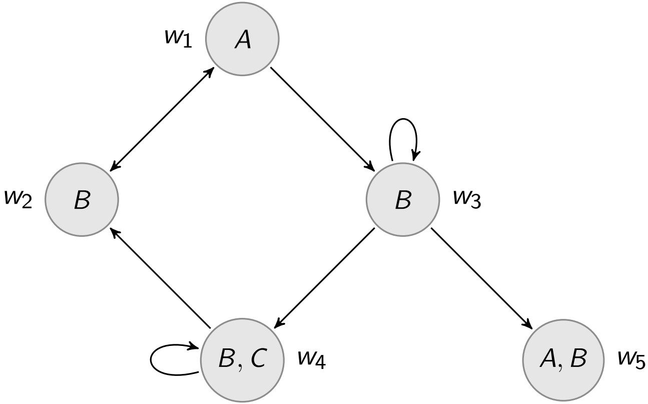
1. Set of states (propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$

Truth: $\mathcal{M}, w \models \varphi$

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$
2. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
5. $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

Example



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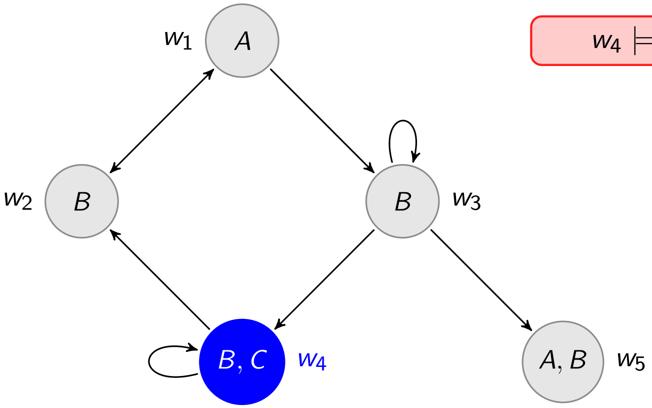
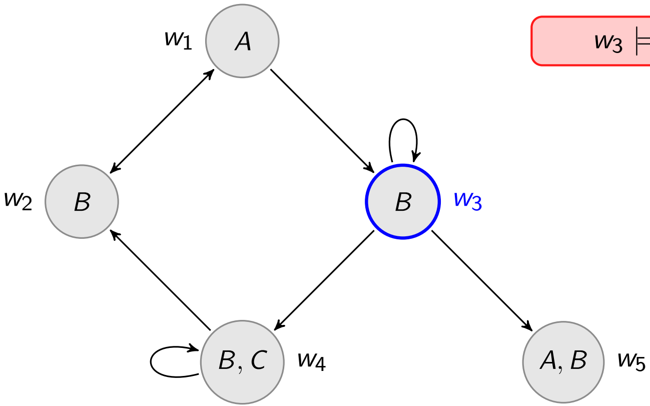


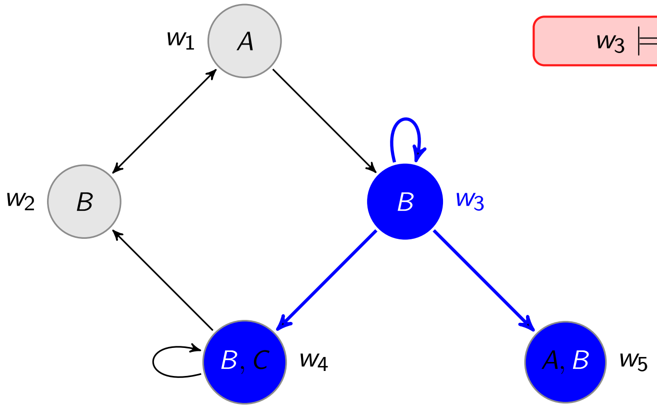
$w_4 \models B \wedge C$

Example



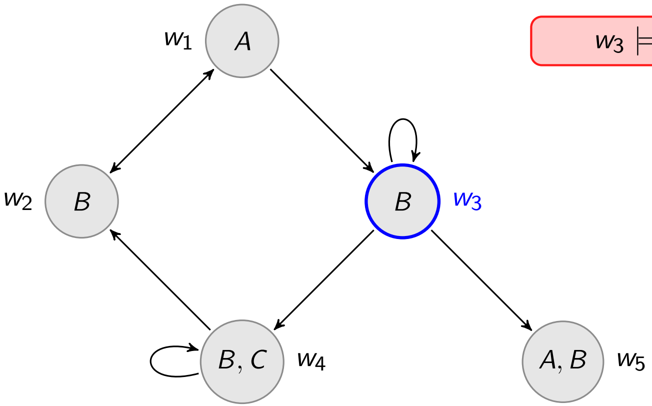
$w_3 \models \Box B$

Example



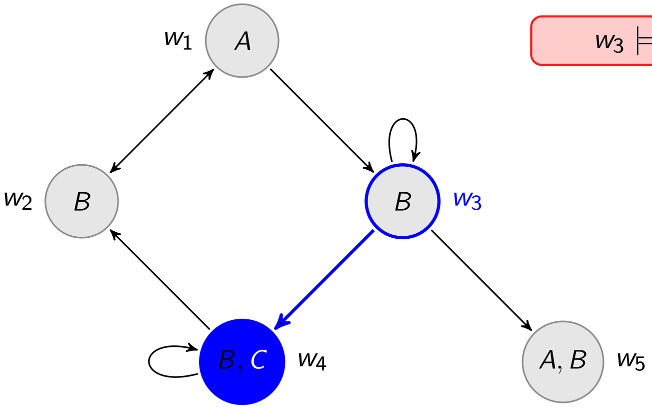
$w_3 \models \Box B$

Example



$w_3 \models \Diamond C$

Example



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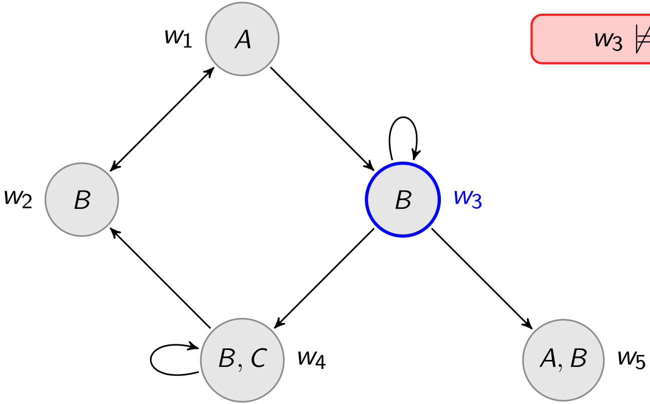
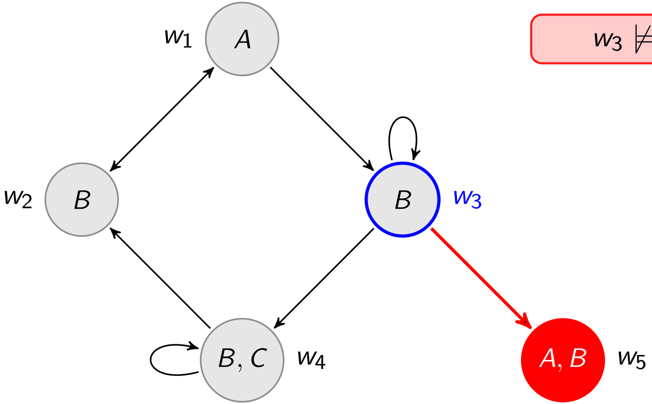


$w_3 \not\models \Box C$

Example



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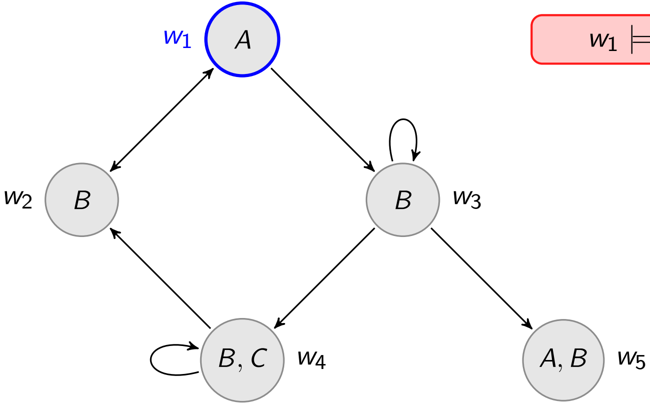
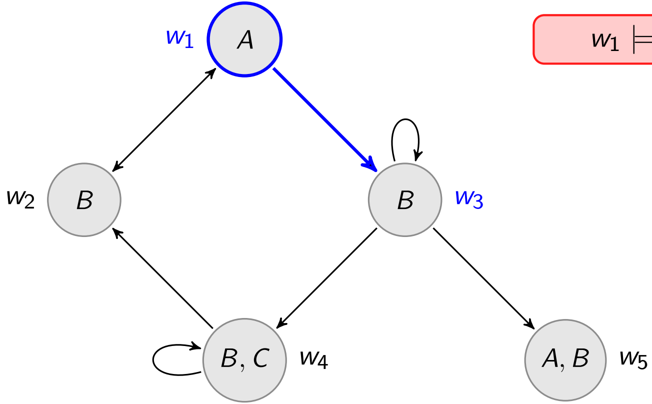


$w_1 \models \Diamond\Box B$

Example



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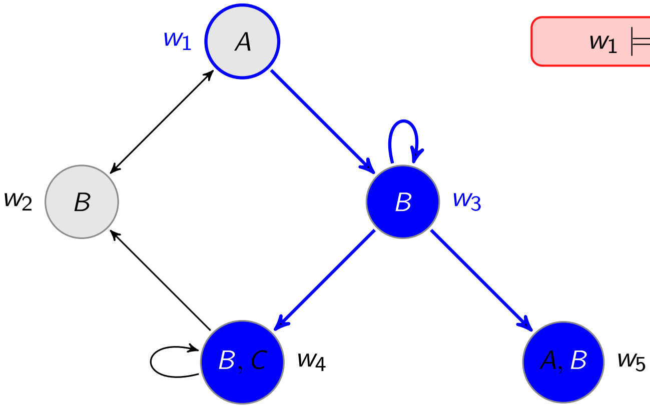

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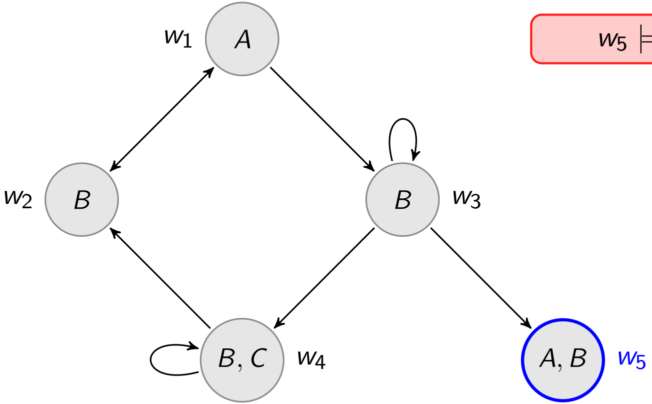
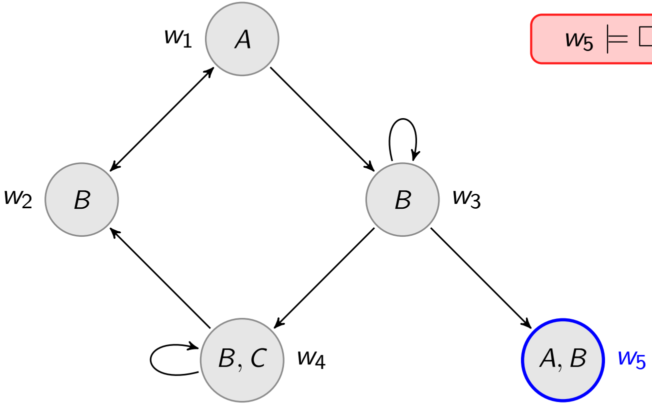


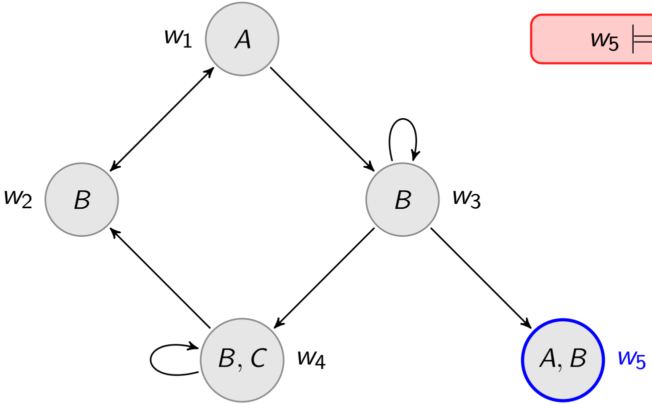
$w_5 \models \Box C$

Example



$w_5 \models \Box(B \wedge \neg B)$

Example



$w_5 \models \neg\Diamond B$

Standard Logical Notions

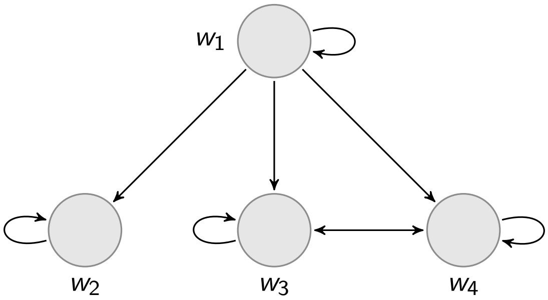
Valid on a model: $\mathcal{M} \models \varphi$

Valid at a state on a frame: $\mathcal{F}, w \models \varphi$

Valid on a frame: $\mathcal{F} \models \varphi$

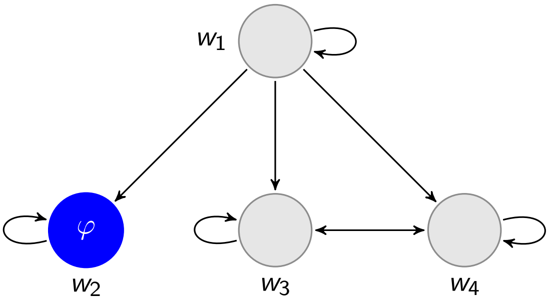
Valid in a class F of frame: $\models_{\mathsf{F}} \varphi$

Example



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \to \Diamond\Box\varphi$$

Example



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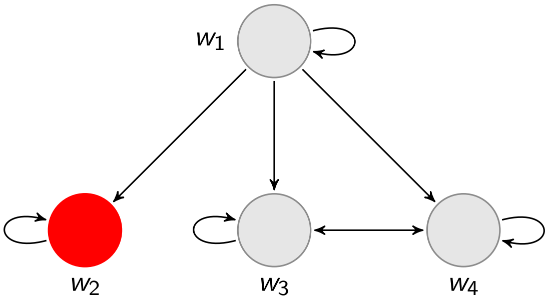

$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \to \Diamond\Box\varphi$$

Some Validities

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
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(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

(RM) $\dfrac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

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What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

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contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.

Truth Sets

Given $\varphi \in \mathcal{L}$ and a model $\mathcal{M}$ on a set of state W, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

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$$[\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$$

$$[\![\cdot]\!]_{\mathcal{M}} : \mathcal{L} \to \wp(W)$$

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

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K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

> *This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.*
>
> (pg. 1)

Neighborhoods in Modal Logic

Neighborhood Frame: $\langle W, N \rangle$

Neighborhood Model: $\langle W, N, V \rangle$

- $W \neq \emptyset$
- $N : W \to \wp(\wp(W))$
- $V : \mathsf{At} \to \wp(W)$

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Two routes to a logical framework

1. Identify interesting patterns that you (do not) want to represent
2. Identify interesting structures that you want to reason about

Key Validities

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
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Logics of High Probability

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

Abilities

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(alternatively, i has the ability to bring about φ)

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4. $Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$

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5. $Abl_i(\varphi \wedge \psi) \rightarrow (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \rightarrow Abl_i\varphi$, $Abl_iAbl_i\varphi \rightarrow Abl_i\varphi$

Games: $(\mathit{Abl}_i\varphi \wedge \mathit{Abl}_i\psi) \not\to \mathit{Abl}_i(\varphi \wedge \psi)$

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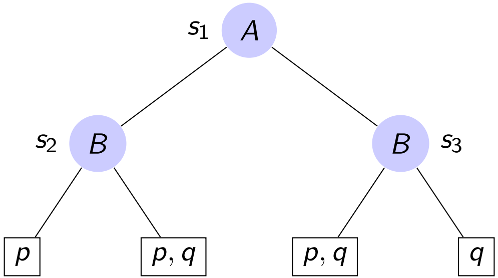
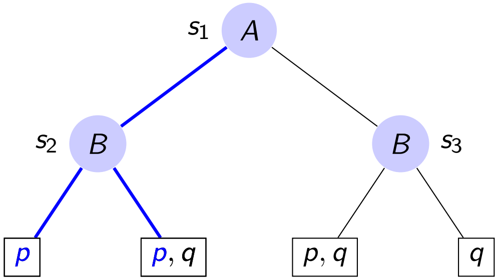
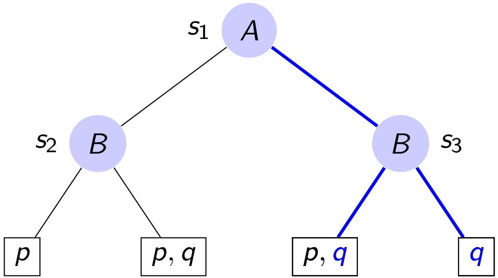


Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



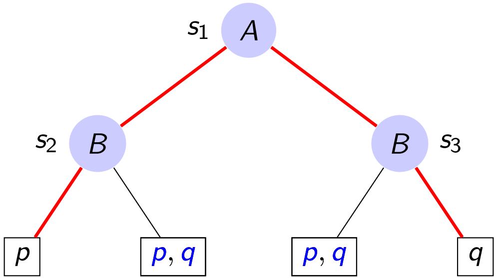
$$s_1 \models Abl_A p$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q \wedge \neg Abl_A(p \wedge q)$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

R. Parikh. *The Logic of Games and its Applications*. Annals of Discrete Mathematics (1985).

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica (2003).

J. van Benthem. *Logic and Games*. Course notes (2007).

Question

$\Box_i\varphi$ means "player i has a strategy to win the game"

$\Diamond_i\varphi$ means "player i's opponent has a strategy to win the game"

Question

$\Box_i \varphi$ means "player i has a strategy to win the game"

$\Diamond_i \varphi$ means "player i's opponent has a strategy to win the game"

- Is $\neg\Diamond_i\neg\varphi \rightarrow \Box_i\varphi$ valid?
- Is $\Box_i\varphi \rightarrow \neg\Diamond_i\neg\varphi$ valid? Hint: the formula is equivalent to $\neg(\Box_i\varphi \wedge \Diamond_i\neg\varphi)$

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

Intuitively, we do not want to say that Ann has the *ability* to throw a bull's eye even though it happens to be true.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, it seems true that Ann does not have the ability to hit the top half of the dart board, and also she does not have the ability to hit the bottom half of the dart board.

Abilities

$Abl_i\varphi$: agent i has the ability to bring about (see to it that) φ is true

What are core logical principles? Depends very much on the intended "application" and how actions are represented...

1. $Abl_i\varphi \rightarrow \varphi$ (or $\varphi \rightarrow Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \rightarrow Abl_i(\varphi \wedge \psi)$
4. $Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$
5. $Abl_i(\varphi \wedge \psi) \rightarrow (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \rightarrow Abl_i\varphi$, $Abl_iAbl_i\varphi \rightarrow Abl_i\varphi$

On the Logic of Ability

$$Abl_i\top$$

$$\varphi \to Abl_i\varphi$$

$$(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$$

$$Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$$

On the Logic of Ability

$\neg Abl_i \top$

$\varphi \not\to Abl_i \varphi$

$(Abl_i \varphi \wedge Abl_i \psi) \not\to Abl_i(\varphi \wedge \psi)$

$Abl_i(\varphi \vee \psi) \not\to (Abl_i \varphi \vee Abl_i \psi)$

On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$\varphi \to \Diamond\varphi$ is valid in the class of reflexive frames

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

On the Logic of Ability

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$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

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$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$\varphi \to \Diamond\varphi$ is valid in the class of reflexive frames

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$ is valid in the class of all frames

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"
(p. 135)

A. Kenny. *Will, Freedom and Power*. 1975.

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"

(p. 135)

A. Kenny. *Will, Freedom and Power*. 1975.

"Even if opportunity only knocks once, I may be able to act on it, and may be culpable for doing so, or for failing to do so."

(p. 1)

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $E\varphi \wedge E\psi \to E(\varphi \wedge \psi)$
4. $E\varphi \to \varphi$
5. $E\varphi \to C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

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Note: the language is restricted so that $\Box\Box\alpha$ *is not a wff.*

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Consensus: α is accepted provided *everyone* accepts α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(C) $(\Box\alpha \wedge \Box\beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(N) $\Box\top$

(D) $\neg\Box\bot$

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(D) $\neg\Box\bot$

Theorem The above axioms axiomatize consensus (provided $n \geq 2^{|\mathsf{At}|}$)).

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Majority: α is accepted if a *majority* of the agents accept α.

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(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(S) $\Box\alpha \rightarrow \neg\Box\neg\alpha$

(T) $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \rightarrow \bigwedge_{1\leq i\leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I$: $|\{i \mid v_(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

Theorem The above axioms axiomatize majority rule.

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Majority: α is accepted if a *majority* of the agents accept α.

Why is $\Box\alpha \wedge \Box\beta \rightarrow \Box(\alpha \wedge \beta)$ invalid?

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Majority: α is accepted if a *majority* of the agents accept α.

Why is $\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$ invalid?

	p	q	$p \wedge q$
i	1	1	1
j	1	0	0
k	0	1	0
Majority	1	1	0

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

T. Daniëls. *Social Choice and Logic via Simple Games*. ILLC, Masters Thesis, 2007.

Logic of Deduction

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Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the set of propositional formulas.

Let $\Sigma \subseteq \mathcal{L}_0$ be the universe

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Interpretation: $(\cdot)^* : \mathsf{At} \to \wp(\Sigma)$

- $(\varphi \vee \psi)^* = (\varphi)^* \cup (\psi)^*$
- $(\neg\varphi)^* = \Sigma - (\varphi)^*$
- $(\Box\varphi)^* = \{\alpha \in \Sigma \mid (\varphi)^* \vdash \alpha\}$

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P. Naumov. *On modal logic of deductive closure*. APAL (2005).

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Fact: $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ is not valid.

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Validities: $\varphi \to \Box\varphi$,

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Validities: $\varphi \to \Box\varphi$, (Mon),

P. Naumov. *On modal logic of deductive closure*. APAL (2005).

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- $(\neg\varphi)^* = \Sigma - (\varphi)^*$
- $(\Box\varphi)^* = \{\alpha \in \Sigma \mid (\varphi)^* \vdash \alpha\}$ (*the deductive closure of* φ)

Validities: $\varphi \to \Box\varphi$, (Mon), $\Box(\varphi \vee \Box\varphi) \to \Box\varphi$

P. Naumov. *On modal logic of deductive closure*. APAL (2005).

Key Validities

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

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(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith

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L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

✓ Jones murders Smith

2. Jones ought not to murder Smith

✓ If Jones murders Smith, then Jones ought to murder Smith gently

4. Jones ought to murder Smith gently

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1. Jones murders Smith
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3. If Jones murders Smith, then Jones ought to murder Smith gently
4. Jones ought to murder Smith gently

$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

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$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently
4. Jones ought to murder Smith gently

✓ If Jones murders Smith gently, then Jones murders Smith.

(Mon) If Jones ought to murder Smith gently, then Jones ought to murder Smith

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L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

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$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

✓ Jones ought to murder Smith gently

5. If Jones murders Smith gently, then Jones murders Smith.

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

7. Jones ought to murder Smith

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L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
- ✗ Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently
4. Jones ought to murder Smith gently
5. If Jones murders Smith gently, then Jones murders Smith.
6. If Jones ought to murder Smith gently, then Jones ought to murder Smith
- ✗ Jones ought to murder Smith

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

~~(M)~~ ~~$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$~~

~~(C)~~ ~~$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$~~

~~(N)~~ ~~$\Box\top$~~

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

~~(Dual)~~ ~~$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$~~

~~(Nec)~~ ~~from $\vdash \varphi$ infer $\vdash \Box\varphi$~~

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

Nec From φ, infer $\Box\varphi$

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
Nec	From φ, infer $\Box\varphi$
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
closure under logical implication

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
closure under known implication

Nec From φ, infer $\Box\varphi$

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *closure under known implication*
Nec	From φ, infer $\Box\varphi$ *knowledge of all logical validities*
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *closure under known implication*
Nec	From φ, infer $\Box\varphi$ *knowledge of all logical validities*
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$ *closure under logical equivalence*

Logical Omniscience/Knowledge Closure

RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
closure under logical implication

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
closure under known implication

Nec From φ, infer $\Box\varphi$
knowledge of all logical validities

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$
closure under logical equivalence

Logical Omniscience/Knowledge Closure

W. Holliday. *Epistemic closure and epistemic logic I: Relevant alternatives and subjunctivism*. Journal of Philosophical Logic, 1 - 62, 2014.

J. Halpern and R. Puccella. *Dealing with logical omniscience: Expressiveness and pragmatics*. Artificial Intelligence 175(1), pgs. 220 - 235, 2011.

Key Validities

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(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If $\mathbf{L}$ is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If $\mathbf{L}$ is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

A modal logic $\mathbf{L}$ is **normal** provided $\mathbf{L}$ is

- contains propositional logic (i.e., all instances of the propositional axioms and closed under Modus Ponens)
- closed under Necessitation (from $\vdash_{\mathbf{L}} \varphi$ infer $\vdash_{\mathbf{L}} \Box\varphi$);
- contains all instances of K $(\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$; and
- closed under *uniform substitution*.

(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If **L** is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

A modal logic **L** is **normal** provided **L** is

- contains propositional logic (i.e., all instances of the propositional axioms and closed under Modus Ponens)
- closed under Necessitation (from $\vdash_{\mathbf{L}} \varphi$ infer $\vdash_{\mathbf{L}} \Box\varphi$);
- contains all instances of K ($\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$); and
- closed under *uniform substitution*.

Normal Modal Logic

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

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Theorem. **K** is sound and strongly complete with respect to the class of all relational frames.

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Theorem. For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

Normal Modal Logic

The smallest normal modal logic **K** consists of

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Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

Theorem. $\mathbf{K} + \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive relational frames.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

Nec $$\frac{\varphi}{\Box\varphi}$$

MP $$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

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RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

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EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

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EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

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E is the smallest classical modal logic.

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EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Are there non-normal extensions of **K**?

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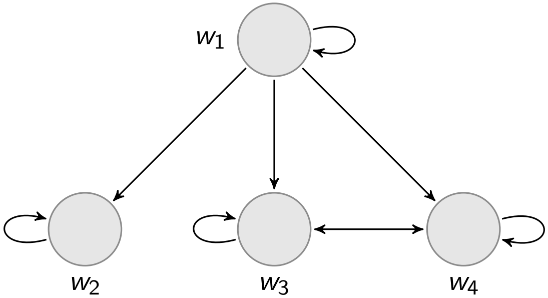
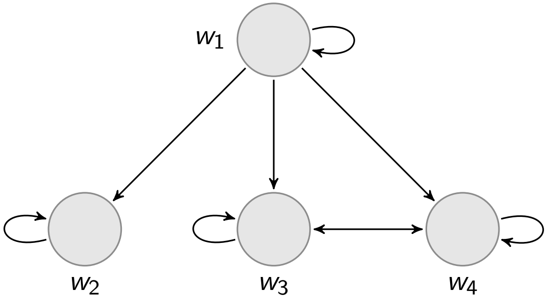
Let **L** be the smallest modal logic containing

- **S4** (**K** $+$ $\Box\varphi \to \varphi$ $+$ $\Box\varphi \to \Box\Box\varphi$)
- all instances of M: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.

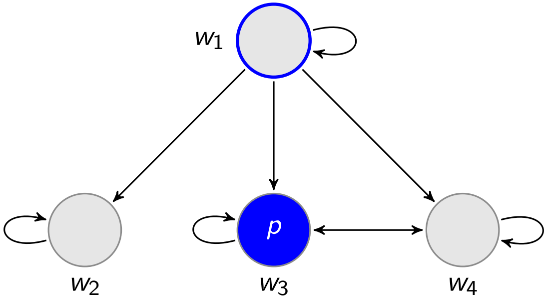


$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \to \Diamond\Box\varphi$$

$$\mathbf{L} \subseteq \mathbf{L}_{w_1} = \{\varphi \mid \mathcal{F}, w_1 \models \varphi\}$$



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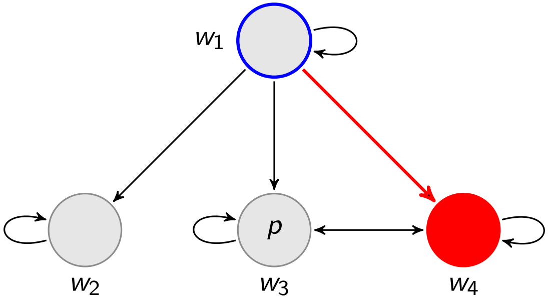
$\mathbf{L} \subseteq \mathbf{L}_{w_1} = \{\varphi \mid \mathcal{F}, w_1 \models \varphi\}$

$\mathcal{F}, w_1 \not\models \Box(\Box\Diamond p \rightarrow \Diamond\Box p)$



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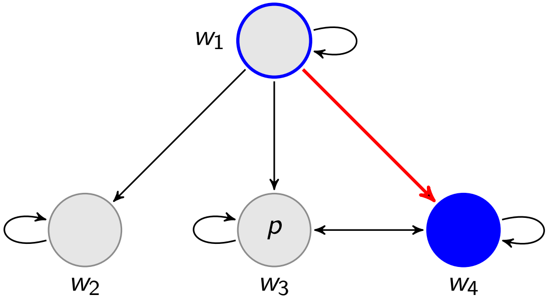

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Plan

✓ Introductory Remarks

✓ Background: Relational Semantics for Modal Logic

✓ Why *Non-Normal* Modal Logic?

- Fundamentals
  - Subset Spaces
  - Neighborhood Semantics
- Why Neighborhood Semantics?

Some Terminology: Subset Spaces

Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i\in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under unions if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i\in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under complements if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

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Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ contains the unit provided $W \in \mathcal{F}$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is proper if $X \in \mathcal{F}$ implies $X^C \notin \mathcal{F}$.
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Lemma

$\mathcal{F}$ is supplemented iff if $X \cap Y \in \mathcal{F}$ then $X \in \mathcal{F}$ and $Y \in \mathcal{F}$.

A few more definitions

- $\mathcal{F}$ is a filter if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.
- $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is a topology if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.

Some Facts

Lemma

If $\mathcal{F}$ is augmented, then $\mathcal{F}$ is closed under arbitrary intersections. In fact, if $\mathcal{F}$ is augmented then $\mathcal{F}$ is a filter.

Fact

There are consistent filters that are not augmented.

Lemma

If $\mathcal{F}$ is closed under binary intersections (i.e., if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$), then $\mathcal{F}$ is closed under finite intersections.

Corollary

If W is finite and $\mathcal{F}$ is a filter over W, then $\mathcal{F}$ is augmented.

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Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N\rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N\rangle$ is a tuple $\langle W, N, V\rangle$ where $V : \mathsf{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \not\in N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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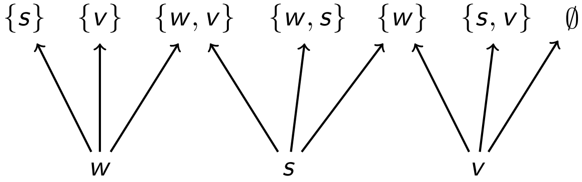

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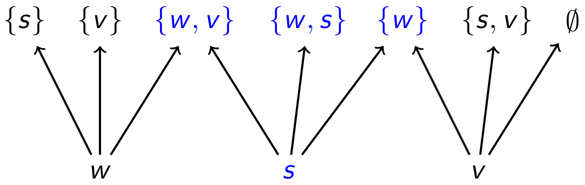

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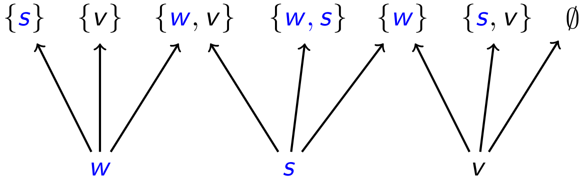
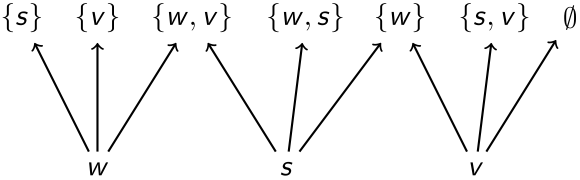


Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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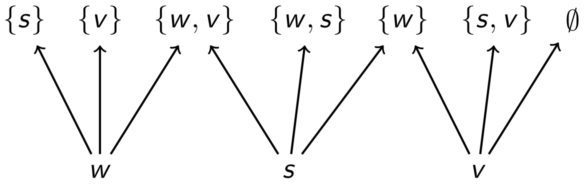
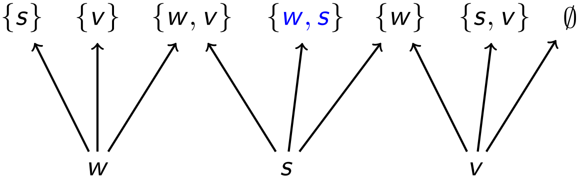


$$\mathfrak{M}, s \models \Box p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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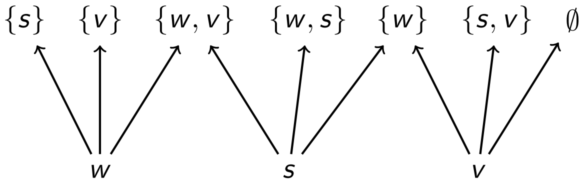

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Detailed Example

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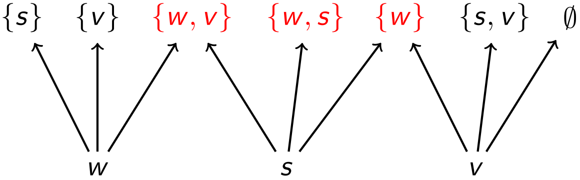
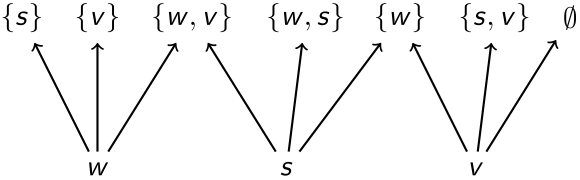


$$\mathfrak{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

$\mathfrak{M}, w \models \Box\Box p$?

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Detailed Example

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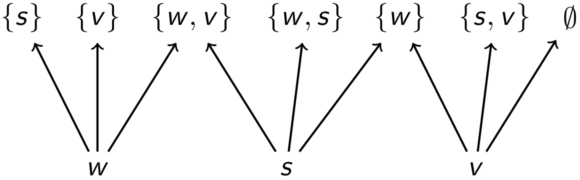

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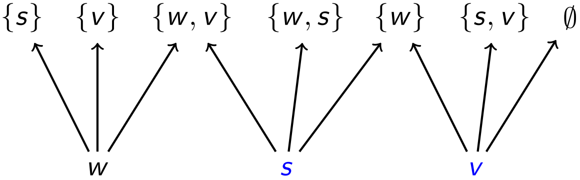

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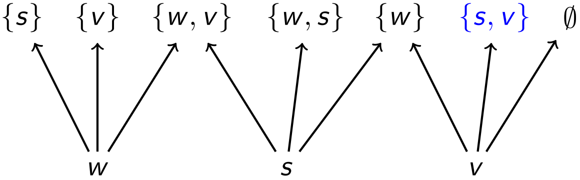

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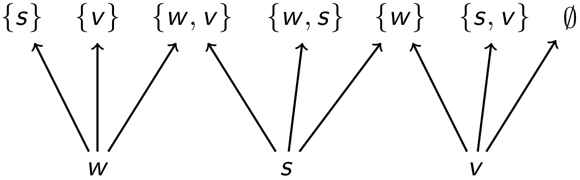

$\mathfrak{M}, w \not\models \Diamond\Box p$

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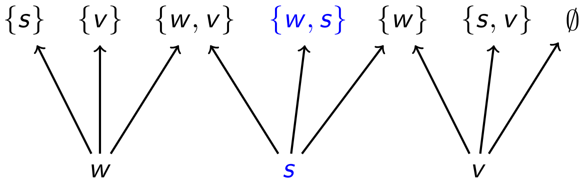
Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$ $\qquad$ $\mathfrak{M}, v \models \Box\Diamond p$

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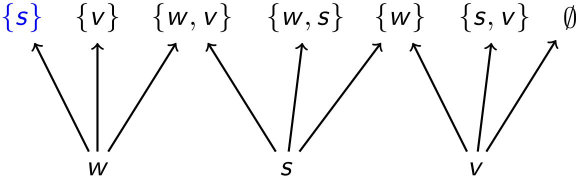

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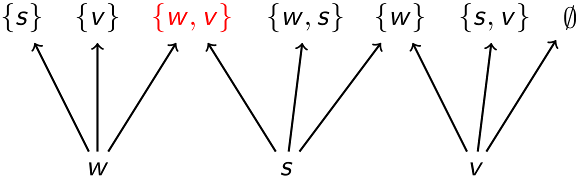

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End of lecture 1